Mathematical Description of the Correctness of Transmission Numerical Codes for Gaussian Channels in the Rough Segregation of Energy

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***Abstract*—Reviewed by the decision optimization problem, minimizing dispersion of errors receive numerical codes in the conditions of the Gaussian fluctuation noise. Formulas to assess the functions likelihood and dispersion errors in different ways signal/noise provided for the position significant bits of numbers. Investigated the opportunity to improve the accuracy at the expense of different distribution vectors amplitude coefficients for each of the transferred numbers. Given an explanation of physical nature of the phenomenon, leading to the differentiation of mean square error recovery numbers of natural a number of transmitted in the same conditions.**

I. Introduction

The processes associated with the processing and transmission of quantitative information are characteristic of most automated control and communication systems, as well as telemetry and radar systems [1]. The qualitative characteristics of management are largely determined by the correctness of the data transmitted and received under the conditions of perturbing environmental factors. Therefore, the relevance of improving the methods and means of rational presentation and efficient processing of numerical data in terms of interference is beyond doubt.

The research results presented in this paper are devoted to clarifying some ideas about the characteristics of numerical systems that use non-uniform energy redundancy as a means of reducing the mean square error of transmitting numbers over Gaussian channels.

The fundamental principles of the theory of numerical data transmission were created by C. Shannon in the framework of the general theory of information [2]. Various information-theoretical aspects of the issues under consideration constitute elements of the theory of the rate-distortion theory, which is being actively developed at the present time [3-6]. Interesting technical applications and features of operating with numerical data in channels with interference using different types of manipulation coding and modulation are described in some detail in [7-11].

This article is a continuation of research aimed at a more in-depth study of the features and phenomena characteristic of the transfer of quantitative information and numerical noise-resistant codes. In particular, the focus is on optimizing the transmission by the criterion of the minimum of the mean square error of restoring numbers when limited to the average transmitter power. Despite the sufficiently traditional nature of the problem (in general, it was considered for a long time [7], [8]), the present work contains a number of original, in the author's opinion, moments related to the in-depth study of the asymmetry of the likelihood functions [12] with an uneven distribution of energy between significant digits of numbers in the binary channel.

The obtained analytical expressions and graphs give a visual representation of the nature of the distortion of numerical codes in Gaussian channels. In this case, a kind of "catalyst" and a source of distortion becomes a phenomenon similar to the "centrifugal increase in error" in the channels with a uniform distribution of energy between the digits of the transmitted numbers.

The purpose of this work is a mathematical description of the process of distorting numerical codes when restoring them at the output of a Gaussian channel, as well as an in-depth study of the consequences of the asymmetry of the likelihood functions for numbers located in different parts of the allowable range under uneven energy protection of positional discharges.

II. Minimizing the Error Variance of the Received Message in the Conditions of Action of the Gaussian Fluctuating Noise

In previous works [9], [10] devoted to the analysis of the probabilistic properties of the transfer of numerical positional codes, it was shown that a simple lexicographic representation of numbers in Gaussian channels using optimal coherent reception of binary signals with phase shift keying causes a systematic error of a special type. The nature of this error is caused by the asymmetry of likelihood functions, which linearly increases as the numbers approach the boundaries of the range on the number axis [13]. Therefore, the distribution of the mean square error of the recovery of numbers has the form of a parabola with a minimum in the center of the range. The nature of the error and the nature of its occurrence allow with good reason to call it centrifugal.

Consider a rather traditional problem is the need to transfer the numbers that are obtained analog-digital conversion of continuous random variable measurements on a channel with Additive white Gaussian noise (AWGN) [14]. With proper coordination of the source and the inverter input channel, messages can be described by a probability distribution $Q(x)$, specified on the set of real numbers $x_0,\dots,x_{m-1}$. Wherein $n=\log_2 m$ – digit numbers $x_i, i=\overline{0,m-1}$, represented in the binary system. We confine ourselves to the most unfavorable (in terms of noise immunity of transmission) case of uniform probability distribution numbers $Q(x_i)=m^{-1}, i=\overline{0,m-1}$, That corresponds to the entropy surplus source to a single number, equally $H_x=m$ bit. In this case, there are no correlations between the numbers of discharges taking with equal probability $q=0,5$ value "0" or "1". In this case, the best interference immunity (maximum likelihood) is achieved in coherent reception chip unit discharges numbers [8, 9]. for any j-th bit using the opposite signal with phase modulation (PM), the probability of receiving the error is determined

$$p(E_j)=\frac{1}{\sqrt{\pi N_0}}\int\limits_{-\infty}^{\sqrt{2E_j/N_0}} e^{-\frac{z^2}{2}}dz, \qquad (1)$$

where $E_j=a_j^2\frac{T}{2}, j=\overline{0,n-1}$ – energy signal with amplitude a_j per modulation interval T, N_0 - power spectral density of the AWGN. Equation (1) corresponds to the standard "hard" decision rules when receiving a single j- the symbol of arbitrary i - th. In a simplified form of the rule represented by the expression

$$x_{i,j}^{*}=\begin{cases}1, s_j\geq a_j;\\ 0, s_j<0;\end{cases} \qquad (2)$$

where $x_{i,j}^{*}$ – evaluation of the value of the discharge binary number,

$$s_j=\frac{2}{T}\int\limits_0^T\left(c_j(t)+\xi(t)\right)\sin\left(\omega_0 t\right)dt, \qquad c_j(t)=\pm a_j\sin\left(\omega_0 t\right),$$

$t\in\overline{0,T}$ (Initial phase is omitted for simplicity), $\xi(t)$ - realization AWGN modulation range T (hereinafter, simply $T=1$), ω_0 – carrier frequency. In the case of uniform distribution of the energy transmitted between the significant digits of numbers $E_j=\text{const}=E$, $a_j=\alpha$, $j=\overline{0,n-1}$, Dispersion error recovery numbers a given positional weight and discharges the formula (1) for receiving a chip unit with independent errors (Binary Symmetric Channel) is defined by the expression

$$D_0=\frac{1}{\sqrt{\pi N_0}}\int\limits_{\alpha}^{\infty} e^{-\frac{z^2}{N_0}}dz\cdot\sum_{j=0}^{n-1}2^{2j}, \qquad (3)$$

One known way to reduce the mean power errors (3) is a redistribution of energy expended in transmitting symbols of numbers, which is performed in accordance with the positional weight significant digits [7]. The average energy of the signals used for the transmission numbers remains fixed, and the variance of the reconstruction error is determined

$$D_1=\frac{1}{\sqrt{\pi N_0}}\cdot\sum_{j=0}^{n-1}2^{2j}\int\limits_{a_j}^{\infty} e^{-\frac{z^2}{N_0}}dz, \qquad (4)$$

where the coefficients $a_j, j=\overline{0,n-1}$ find solutions conditional minimization problem

$$\min_{a_j}\langle D_1\rangle, \sum_{j=0}^{n-1}a_j^{\,2}=n\cdot\alpha^2. \qquad (5)$$

Implementation of the "soft" algorithm of reception of a single discharge interval presupposes $\left[-a_j,a_j\right]$, Inside which the evaluation of the received receiver discharge is treated as a continuous quantity, varying within the respective positional weight j-th bit $\left[0,2^j\right]$. The corresponding decision rule has the form

$$x_{i,j}^{*}=\begin{cases}1, & s_j\geq a_j;\\ \frac{1}{a_j}s_j, & \left|s_j\right|<a_j;\\ 0 & s_j\leq -a_j.\end{cases} \qquad (6)$$

This leads to the following definition dispersion numbers error recovery

$$D_2=\sum_{j=0}^{n-1}\left\{\frac{2^{2j}}{\sqrt{\pi N_0}}\left[\int\limits_0^{2a_j}\left(\frac{z}{2a_j}\right)^2 e^{-\frac{z^2}{N_0}}dz+\int\limits_{2a_j}^{\infty} e^{-\frac{z^2}{N_0}}dz\right]\right\}. \qquad (7)$$

Coefficients a_j, minimizing dispersion D_2, are solutions of the problem, similar to expression (5):

$$\min_{a_j}\langle D_2\rangle, \sum_{j=0}^{n-1}a_j^{\,2}=n\cdot\alpha^2. \qquad (8)$$

Hybrid "semi-soft" algorithm reception numbers obtained from the previous administration by changing the interval boundaries, within which the evaluation values of the discharge receiver is perceived as a continuous quantity. Rule evaluation represented by the expression

$$x_{i,j}^{*}=\begin{cases}1, & s_j\geq(1-b_j)\cdot a_j;\\ \frac{1}{a_j}s_j, & \left|s_j\right|<(1-b_j)\cdot a_j;\\ 0 & s_j\leq(1-b_j)\cdot a_j.\end{cases}$$

Coefficients $b_j, j = \overline{0, n-1}$ can take values from the range from 0 to 1. When $b_j = 0$ or $b_j = 1$ We have, respectively, previously considered a "soft" or "hard" decision-making algorithms for the j-th day of discharge. The formula for calculating the error variance of reception numbers obtained based on the expressions (7) and has the form

$$D_3 = \frac{1}{\sqrt{\pi N_0}} \sum_{j=0}^{n-1} \left\{ 2^{2j} \left[\int_{b_j a_j}^{(2-b_j)a_j} \left(\frac{z}{2a_j} \right)^2 e^{-\frac{z^2}{N_0}} dz + \int_{(2-b_j)a_j}^{\infty} e^{-\frac{z^2}{N_0}} dz \right] \right\}, \tag{9}$$

where the sets of coefficients a_j and b_j solves the problem of conditional minimization of the form

$$\min_{a_j, b_j} \langle D_3 \rangle, \sum_{j=0}^{n-1} a_j^{\,2} = n \cdot \alpha^2, 0 \le \underset{j=\overline{0,n-1}}{b_j} \le 1. \tag{10}$$

Obviously, a hybrid algorithm (8), (9) is the most versatile because it implies a separate optimization of the parameters of the decision rule reception bits numbers. Under the conditions that determine the minimum of the objective function (10), the decision criterion for each item number may vary with different values of channel quality is not dependent on the aggregate discharges other admission criteria.

For practical guidance on the synthesis of optimal numbers of transmitting and receiving positional algorithm, codes obtained numerical solutions of (5), (8) and (10) for different values of the signal/noise in channels with AWGN.

A brief illustration of the results of solutions obtained for the transmission and processing algorithms eight bit binary number (byte) shown in Fig. 1-3.

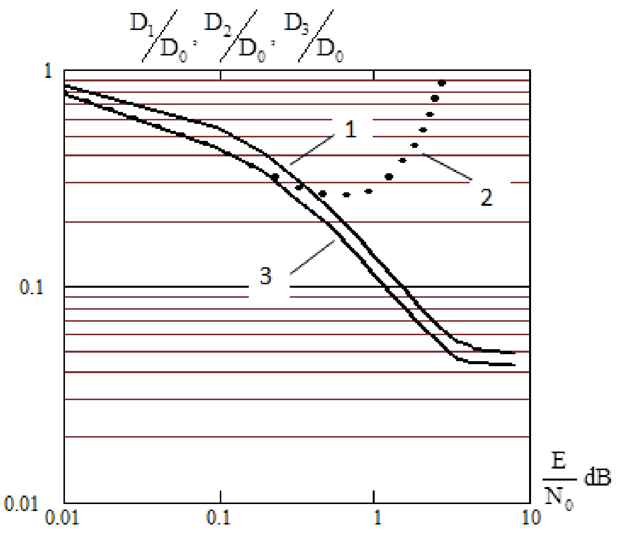


Fig. 1. Dependence of normalized average power value of the error signal/noise ratio

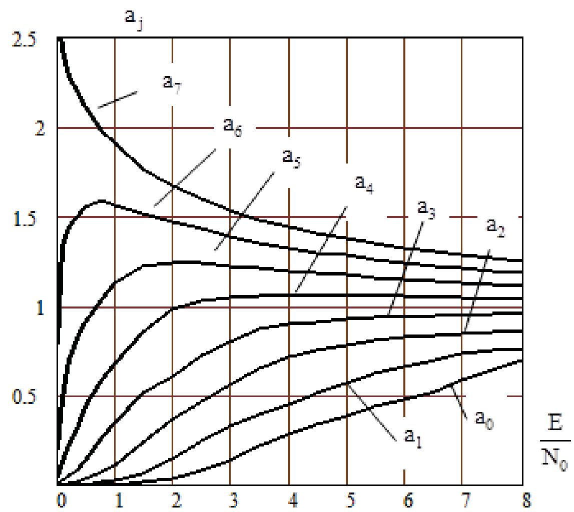


Fig. 2. Dependence of the amplitude of the coefficients of the transfer of bytes from the signal/noise ratio

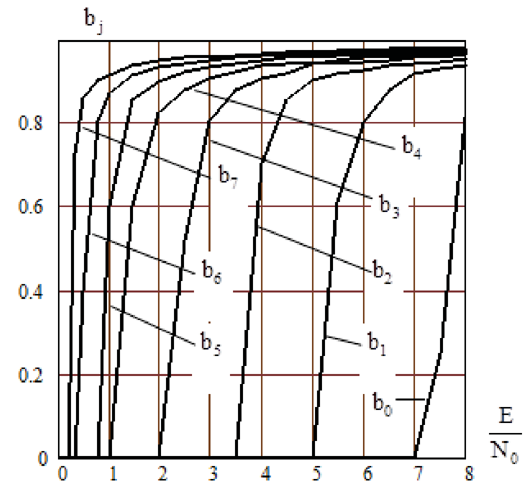


Fig. 3. The dependence of the boundary coefficients "semi-soft" algorithm of the signal/noise ratio

Figure 1 shows the normalized value D_0 (See. the expression (3)), depending on the average power errors for the considered three types of decision-making algorithms. When any signal/noise ratio combined algorithm (curve 3) provide the lowest noise variance recovery. For poor channel at $E/N_0 \le 0{,}5$ accuracy combined algorithm coincides with the accuracy afforded by "soft" decisions (6). The good channels with $E/N_0 > 8$ the accuracy of the combined metrics (8) and "hard" (2) algorithms asymptotically coincide.

Figure 2 illustrates the variation of the amplitude of the coefficients a_j Characterizing energy distribution among discharges bytes. In bad channels ($E/N_0 \le 1$) Smallest error power is provided when transferring from 1 to 4 MSBs of the positional number. With channel quality improvement increases the degree of importance of the LSBs, and, accordingly, increases the proportion of energy used for their transmission. Shortened bit numbers transmitted in relations $E/N_0 \le 3$ (Since the lower bits can simply not transmit), which, in fact, does not result in loss of precision, it means having the possibility to reduce the required bandwidth and the maximum transmitter power compared to the total transmission. This follows from the possibility of redistribution of energy by increasing the duration of the modulation interval T Allotted to the transmission of each of the digits of the number.

Interesting is the "behavior" of the boundary coefficients b_j, determining decision algorithm when receiving the individual bits of the numbers (Fig. 3). As follows from the data shown in Figures 1 and 3, the work in bad channels is more efficient in the use of "soft" decisions when processing of all digits of the number ($b_j \to 0,\ j = \overline{0,7}$). When the channel is observed consistent improvement abrupt increase coefficient values b_j, which takes place, starting with the most significant bits. It should be noted that the signal/noise ratio in the range of relevance to the real channels relations (1÷10 dB) nature of the decision rule is stored combined. Processing MSBs produced by the rule, which is close to the "hard" algorithm (2), while the decision on the values of the least significant bits is carried on the "soft" algorithm (8).

We now show that the use of methods of non-uniform energy protection of significant discharges leads to a significant change in the behavior of the mean square error within the range of transmitted numbers. Under conditions of limited average transmitter power, the energy expended in transmitting each of the binary numbers with a constant number of significant digits should remain fixed. However, within the same code word, it is advisable to redistribute the power (amplitude) of the carrier frequency in accordance with the weight of the corresponding positional digits of the transmitted number [8, 10]. The algorithm for forming a sequence of FM signals by a modulator, or more precisely, the distribution of the amplitudes of the carrier oscillations under the conditions of a "hard" decision rule by a demodulator, is determined by solving an optimization problem [15]. The formulation of this problem for n - bit binary number implies finding the vector of amplitude coefficients $\vec{A} = \{a_0, \ldots, a_{n-1}\}$

$$Tg = \min_{\vec{A}} \left\langle \delta_c^2 \right\rangle, \tag{11}$$

ensuring the minimum of the objective function

$$\sum_{i=0}^{n-1} \left(a_i\right)^2 = n. \tag{12}$$

In this case, the minimization function in (11) is the average (over the range) square of the recovery error of the numbers transmitted over the Gaussian channel:

$$\delta_c^2 = \sum_{i=0}^{n-1} 2^{2i-1} \operatorname{erfc}\left(\frac{a_i}{\sqrt{N_0}}\right), \tag{13}$$

Where N_0 – the spectral density of white Gaussian noise. Task (11) - (13) implies an equal probability of the occurrence of all numbers at the transmitter input from a given limited range and is easily solved numerically using the built-in procedures of the MathCAD system. Table I presents the optimization results obtained for a byte (8-bit) number for several values of the signal-to-noise ratio (for fixed average energy spent on the transmission of bits).

TABLE I. A a_i WITH OPTIMAL ENERGY DISTRIBUTION

E/N_0 \ i	0	1	2	3	4	5	6	7
1	0,004	0,018	0,070	0,264	0,696	1,165	1,565	1,907
2	0,026	0,103	0,335	0,675	0,981	1,240	1,465	1,666
3	0,103	0,315	0,592	0,835	1,042	1,223	1,384	1,531
4	0,265	0,505	0,717	0,897	1,054	1,194	1,322	1,440
5	0,423	0,617	0,782	0,924	1,052	1,167	1,274	1,373

The main way to describe random transformations of transmitted numbers is to find likelihood functions. f(Y|k), giving the probability distribution of the channel output at a fixed input state $k=0,\ldots,2^n-1$. To do this, we define the embedded vector of Hamming's mutual distances between codewords corresponding to numbers in - dimensional space:

$$\vec{\Theta} = \left(\left(\theta_m\right)_j\right)_i, \quad m \in \overline{0, n-1}; \quad i, j \in \overline{0, 2^n - 1}.$$

The elements of this vector are Boolean variables and position the differences between the bits of the same name of the binary representations of the valid range:

$$\left(\left(\theta_m\right)_j\right)_i = \left(b_i\right)_m \oplus \left(b_j\right)_m, \tag{14}$$

Where b_i, b_j – binary representation of numbers and, respectively; m – the value of the first bit (0 or 1) of the number. The formalization of the set of mutual distances (14) makes it possible to determine the likelihood function corresponding to the transmitted number k as a string containing the probabilities of the corresponding transitions

$$f\left(Y|k\right) = \left\{f_0\left(Y|k\right), \ldots, f_{2^n-1}\left(Y|k\right)\right\}, \tag{15}$$

where

$$f_j\left(Y|k\right) = \prod_{i=0}^{n-1}\left[p_i^{\left(\left(\theta_k\right)_j\right)_i}\right] \cdot \prod_{m=0}^{n-1}\left[\left(1-p_m\right)^{\left[1-\left(\left(\theta_k\right)_j\right)_m\right]}\right] \tag{16}$$

$p_i = \frac{1}{2}\operatorname{erfc}\left(\frac{a_i}{\sqrt{N_0}}\right), i = \overline{0, n-1}$ – reception probability with the error of the corresponding number bits.

The obtained formula (16) is an analogue of the binomial distribution law (Bernoulli) for the case of non-improbable bit distortion in the Gaussian channel with uneven energy distribution.

The calculations of the likelihood functions using expressions (15) and (16), make it possible to analyze the properties of the channel distorting action. In figure 4 shows several examples of functions calculated f(Y|k) for a "bad" channel with a signal-to-noise ratio equal to one.

The figure shows the likelihood functions of neighboring numbers from different parts of the range when transmitting a byte number. This is done in order to emphasize the significant (discontinuous) changes in type f(Y|k) occurring in the range of values k that coincide with the integer power of the number "2" - the base of the number system when representing the channel code of numbers.

From the data analysis in fig. 4, it follows that all the likelihood functions, to one degree or another, are asymmetric. This property causes a shift in the mathematical expectation of functions from a magnitude, which, in turn, is a source of systematic error when restoring numbers at the channel output.

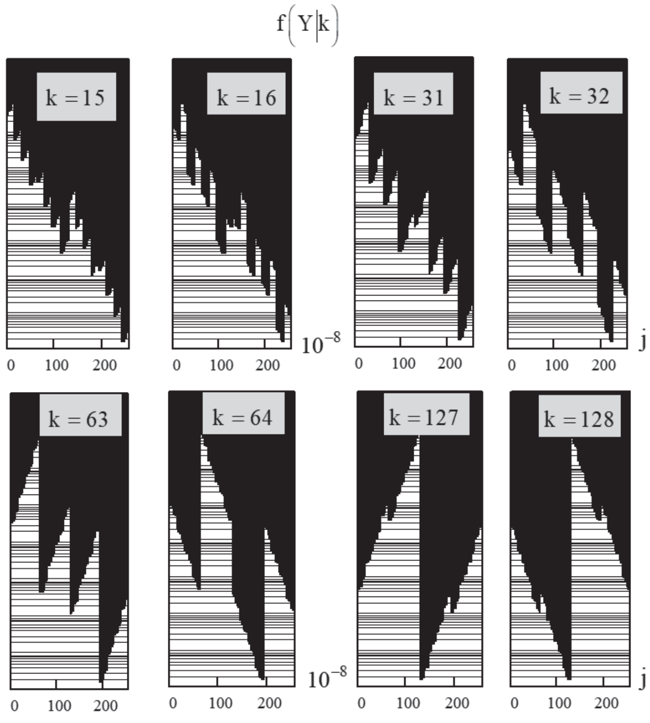


Fig. 4. Likelihood functions with uneven distribution of energy between bits of numbers

It is obvious that the mathematical expectation of a systematic error (estimation bias) can be determined for each number using the expression.

$$M_k = \sum_{i=0}^{2^n-1} i \cdot f_i\left(Y\middle|k\right) - k \cdot \quad (17)$$

In this case, the average square of the error of the numbers is

$$D_k = \sum_{i=0}^{2^n-1} \left(k-i\right)^2 f_i\left(Y\middle|k\right) \cdot \quad (18)$$

The variance of the likelihood functions, which can be calculated by taking into account the estimated bias (17), is the same for any and characterizes the minimum attainable value of the mean square error

$$D_{min} = \sum_{i=0}^{2^n-1} \left(k-i+M_k\right)^2 f_i\left(Y\middle|k\right) \cdot \quad (19)$$

In Figure 5 shows the dependences of the estimated bias (17) (Fig. 5a) and the average powers of the recovery error (18) and (19) (Fig. 5b) on the value of the byte number under conditions similar to Fig. 4.

Analysis of dependences M_k and D_k suggests that the bias estimates of numbers, which appear periodically on the number axis, lead to a significant increase in the value of the mean square error. Given the non-equivalent conditions for the transfer of different numbers in the presence of estimates bias, a small decrease D_k and $\delta_c{}^2$ can be achieved by formulating the problem of optimizing the energy redistribution for each of the binary code words (numbers) separately. In this case, the objective function is

$$\underset{k=0\ldots 2^n-1}{Tg_k} = \min_{\vec{B}} \left\langle \delta_k{}^2 \right\rangle \qquad \text{if} \qquad \vec{B} = \left\{b_0, \ldots, b_n\right\} \quad (20)$$

and limiting condition

$$\left|\vec{B}\right|^2 = n \quad (21)$$

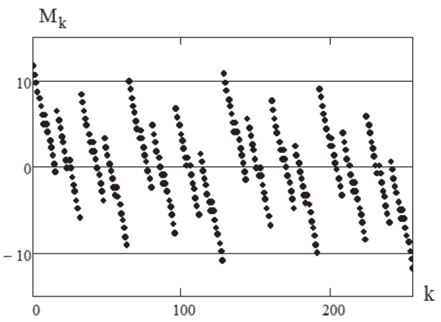
The minimization function in (20) is defined

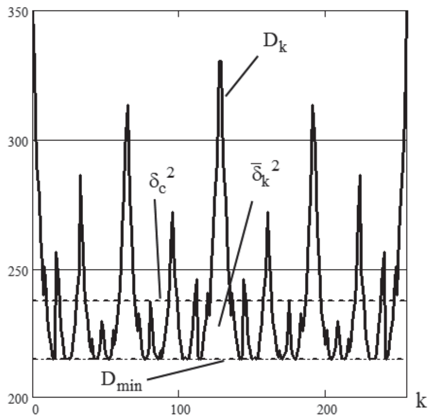
$$\delta_k{}^2 = \sum_{m=0}^{2^n-1} \left\{ \left(m-k\right)^2 \cdot \prod_{i=0}^{n-1} \left[\frac{1}{2}\operatorname{erfc}\left(\frac{b_i}{\sqrt{N_0}}\right)\right]^{\left[\left(\theta_k\right)_m\right]_i} \times \prod_{j=0}^{n-1} \left[1-\frac{1}{2}\operatorname{erfc}\left(\frac{b_j}{\sqrt{N_0}}\right)\right]^{1-\left[\left(\theta_k\right)_m\right]_j} \right\} \quad (22)$$

Thus, the optimization of the energy distribution requires solving problems (20) - (22) for each number 2^n from the permissible range. In this case, the achieved value of the average over the range of the square error is

$$\bar{\delta}_k{}^2 = 2^{-n} \sum_{k=0}^{2^n-1} \delta_k{}^2 \quad (23)$$

The value δ_k^2 calculated for a single signal-to-noise ratio is also shown in Fig. 5b. The solution of problem (20) - (23) gives a certain increment in the accuracy of the transmission of numbers, monotonously increasing with the improvement of the quality of the channel. However, in the modulator's memory, there should be a vector of amplitude coefficients for each of the 2^{n-1} numbers (for half of the total, since the values of the vectors are mirror-symmetric with respect to the 2^{n-1} number).





a) offset estimates for different numbers k;

b) actual D_k, average δ_c^2 and minimum squared errors D_{min};

Fig. 5. Indicators of the accuracy of the recovery numbers for the Gaussian channel with an uneven distribution of energy between the bits of the binary code

The analysis shows that the value δ_k^2 is close to the maximum achievable with a uniform distribution of numbers at the channel input and the realization of reception according to the maximum likelihood rule. The use of closed manipulation codes (gray code and binary minimized code [7, 10, 15]) to represent the numbers in the channel does not improve the overall result.

It should be noted that solving the optimization problem (20) - (22) for each of the numbers separately results in obtaining the optimal values of the amplitude coefficient vectors, for which, for a small signal-to-noise ratio, the "degenerate" (zero) amplitudes correspond to the average weight to discharges, while low-order discharges are not degenerate. Optimization "on average" (11) - (13) under similar conditions leads to vanishing amplitudes of signals corresponding only to the lower bits of the numbers.

III. Conclusion

The asymmetry of the likelihood functions is a fatal phenomenon when transmitting without redundant numeric codes, resulting in different accuracy of recovery of positional numbers located on different parts of the numerical axis. This phenomenon should be taken into account in order to correctly estimate the expected accuracy of transmission over interference channels. The method of optimal distribution of the modulator energy between the digits of numbers allows significantly (by a factor of 7–20 at different channel quality) to reduce the mean square of the number recovery error and reduce the effect of the asymmetry of the likelihood functions. At the same time, the solution of the optimization problem separately for each transmitted number gives an increase in the average recovery accuracy, but requires a larger amount of modulator memory. The subject of further research may be processing methods in the demodulator with input distributions of numbers other than uniform.

Acknowledgment

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