Grayscale and Color Basis Images

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***Abstract*—Orthogonal transformation of digital images can be represented as a decomposition over basis matrices or basis images. Grayscale and color basis images are introduced. For particular case of DWT (Discrete Wavelet Transform) obtained basis wavelet images have a block structure similar to frequency bands of the the DWT coefficients. A steganographic scheme for frequency domain watermarking based on this representation is considered. Presented example of detection algorithm illustrates how this representation can be used for frequency embedding techniques.**

I. Introduction

A digital image has various representations and some of them are required by applications. Many useful representations are produced by orthogonal transforms that are powerful tools of image processing. Well known examples are JPEG and JPEG2000 lossy compression formats based on DCT (Discrete Cosine Transform) and DWT (Discrete Wavelet Transform). For the image compression problem block based DCT and DWT techniques are developed [1]–[4] and generalized to non-separable transforms [5], [6] and irregular transforms [7]–[9].

Orthogonal transform produces scattering of digital data, a process that redistributes pixel energy of transformed image. It is useful for protection of hidden data in steganography, when a message is embedded into image. Hidden data are scattered among all digital cover image and become more robust to lossy data compression and some statistical attacks [10], [11].

The orthogonal transform of images may be considered as a decomposition over matrices known as basis matrices [12]. Being some kind of grayscale images, the basis matrices look attractive and they are often reproduced by textbooks [13]. We will also call these matrices basis images. In this paper we focus on the following items: color and wavelet basis images, orthogonal transform by matrix of basis images and their quantum analogues. For color images the solution is directly achieved by considering three-dimensional orthogonal transform but for wavelets the solution is not so simple. The reason is that in practice, DWT is calculated by algorithms using signal processing techniques instead of orthogonal transforms. Nevertheless these algorithms can be used to calculate wavelet basis images. So it was found for various wavelets that the basis has a block structure similar to DWT coefficients [14], [15].

The watermarking schemes, which use wavelet orthogonal transform, are also available. The technique uses two orthogonal transformations, for example, DCT-SVD [16]–[18]. SVD (Singular Value Decomposition) is conversion of a rectangular matrix into a block-diagonal matrix. Embedding can be performed using the coefficients of the orthogonal SVD transform, which operates with frequency wavelet blocks *LL*, *LH*, *HL*, *HH*. DWT transform is not necessarily one-level [19], [20]. A useful calculation tool is IWT (Integer Wavelet Transform) [21]. In this transform the accuracy of calculations is limited, then the brightness values of the image pixels and transform coefficients are recorded in the same integer encoding. This eliminates the loss of rounding and creates reversibility at least in part of integer encoding.

The paper is organized as follows. Firstly, the orthogonal transformation and decomposition over the basis image are considered. Then we introduce grayscale basis images particulary for wavelets and present an example of detection algorithm for the case of DWT coefficient watermarking. Finally, we discuss RGB basis images.

II. Basis Images

Orthogonal transform of digital image is performed with an orthogonal matrix.

Orthogonal matrix

A matrix of real numbers is said to be *orthogonal* if (for details see [22])

$$UU^T = I,$$

where I is the identity matrix. It implies that

$$U^TU = I.$$

Columns of this matrix u_m and rows u_n^T are orthonormal vectors

$$\langle u_m, u_n\rangle = \delta_{mn},$$
$$\langle u_m^T, u_n^T\rangle = \delta_{mn},$$

where $\langle x, y\rangle$ denotes scalar product of two vectors x and y, and δ_{mn} is the Kronecker symbol.

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Representation of matrix

Let $F = \{F_{mn}\}$ be a real rectangular $M \times N$ matrix, that corresponds to a grayscale image. We introduce two orthogonal matrices $U = \{U_{mn}\}$ and $V = \{V_{pk}\}$ of the size $M \times M$ and $N \times N$ respectively. Then taking into account that the matrix $F = UU^T FVV^T$, we find

$$\begin{aligned} F &= UGV^T, \\ G &= U^T FV, \end{aligned} \tag{1}$$

where G is a $M \times N$ matrix.

Let us assume that F is an image in a spatial domain (that is the image as we see it). Matrix G is usually called a *frequency representation* of F or an image in frequency domain. The frequency domain image may look senseless, however the orthogonal transform is reversible and the original image can always be retrieved.

Using the matrix form of the representation (1)

$$F_{xy} = \sum_{k,p} U_{xk} G_{kp} V^T_{py} = \sum_{k,p} (u_k \otimes v_p)_{xy} G_{kp},$$

we get a decomposition over tensor products of rows and columns of the matrices U and V, denoted by $\otimes$. Thus, if u_k is column vector $u_k = (U_{1k}, U_{2k}, \ldots, U_{Mk})^T$ and $v_p = (V_{1p}, V_{2p}, \ldots, V_{Np})^T$, then $u_k \otimes v_p := u_k v_p^T$ and $(u_k \otimes v_p)_{xy} = U_{xk} V_{yp}$.

Here and later we assume $U = V$ and $M = N$, as this is more interesting case. Then the decomposition produced by the orthogonal transformation takes the form

$$\begin{aligned} F &= \sum_{k,p} (u_k \otimes u_p) G_{kp}, \\ G &= \sum_{x,y} (u^T_x \otimes u^T_y) F_{xy}. \end{aligned} \tag{2}$$

Grayscale basis images

We introduce the matrices

$$a_{kp} := u_k \otimes u_p,$$

$$d_{xy} := u^T_x \otimes u^T_y,$$

that we call *basis images*. There are N^2 basis images of size $N \times N$, every image pixel is a product of two items of the orthogonal matrix U:

$$a_{kp}(x, y) = U_{xk} U_{yp}.$$

Properties of basis images

Being the tensor products of columns and rows of orthogonal matrix, the basis images have properties that follow from orthogonality. We focus on the basis images a_{kp}, as for $U = V$ the properties of d_{xy} are the same.

1) The matrix product of two basis images is another basis image

$$a_{kp}\, a_{mn} = a_{kn} \delta_{pm}.$$

2) The scalar product

$$\langle a_{kp}, a_{mn} \rangle = \delta_{km} \delta_{pn},$$

where the scalar product of matrices is denoted by $\langle A, B \rangle := \sum_{m,n} A_{mn} B_{mn}$.

3) The sums of the diagonal elements are

$$\sum_k a_{kk} = I,$$

$$\sum_k a_{kk}(x, y) = \delta_{xy},$$

$$\sum_x a_{kp}(x, x) = \delta_{kp}.$$

Analyzing these properties we came to the conclusion that the basis images are orthonormal. This observation allows us to consider the orthogonal transform (2) as a standard decomposition over the orthonormal basis. It is obvious that the first equation in (2) takes the form

$$F = \sum_{k,p} G_{kp}\, a_{kp}, \tag{3}$$

where $G_{kp} = \langle F, a_{kp} \rangle$.

A. Generation of basis images

There are at least two ways to get basis images. The first is to use its definitions. In this case the orthogonal matrix has to be given. The second way follows from orthogonal transform of the basis images.

Let us focus on the second approach. Let $F = a_{ab}$ in the representation (3). Then we find the basis image representation of the form $G_{kp} = \delta_{ka} \delta_{pb}$. It means that the matrix G has one non-zero pixel, it is equal to 1 and its position is (a, b). So, the orthogonal transform of a basis image is a binary matrix of unit brightness. We denote such unit matrix as

$$e_{ab} = \{\delta_{ka} \delta_{pb}\},$$

where $k, p = 1, \ldots, N$. Then the following relations are valid

$$\begin{aligned} a_{ab} &= U e_{ab} U^T, \\ d_{ab} &= U^T e_{ab} U. \end{aligned} \tag{4}$$

So together with the unit vectors e_k the unit matrices e_{ab} form a standard basis and the orthogonal transform of the basis is a set of basis images a_{ab}. Indeed, with the help of the standard basis any matrix can be presented in the following form

$$G = \sum_{k,p} G_{kp} e_{kp}.$$

Then we get the decomposition given by (3), using the orthogonal transform and taking into account (4).

Example of WHT basis images

The 2×2 orthogonal Walsh-Hadamard Transform (WHT) matrix known also as Hadamard matrix consists of $+1$ and -1,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

In optics this matrix describes so called 50% beam splitter, a linear optical element often used in experiments to split the beam into two parts. Four basis images a_{kp}, denoted as tensor product of columns, have the following form

$$a_{11} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \qquad a_{12} = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix},$$

$$a_{21} = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}, \quad a_{22} = \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

The determinant of every matrix equals to 0 and the matrices are non invertible. The matrices can be generated from a unit matrix by WHT:

$$H: \quad e_{11} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \leftrightarrows \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = a_{11}.$$

This equation illustrates relations between the basis images and the standard two-dimensional basis. But what is more interesting, the equation demonstrates scattering of digital data. So, a non-zero pixel of the unit matrix transforms into a basis images of a matrix with only non-zero pixels.

As a result, basis images can be produced by transformation of unit matrices.

The quantum analogue

The presented features allow us to consider basis images as a representation of quantum operators. These operators describe transitions of a physical system between its states or levels. We use here the quantum mechanics notation, details see in [23].

Let us assume that $\{|k\rangle\}$ and $\{|q\rangle\}$ are two basis of a single particle Hilbert space

$$\sum_k |k\rangle\langle k| = 1,$$
$$\sum_q |q\rangle\langle q| = 1,$$

where $k \in Z = \{1, 2, \dots\}$, $q \in Q = \{x, y, \dots\}$. Let the overlapping integrals be real

$$\langle k|q\rangle^* = \langle q|k\rangle. \tag{5}$$

Then we find a real matrix $\widetilde{U}_{qk} = \langle q|k\rangle$ that is orthogonal, because Z and Q are complete basis.

The following operator

$$|k\rangle\langle p| = \widehat{a}_{kp}, \tag{6}$$

where $k, p \in Z$, describes transition from the state or level $|p\rangle$ into level $|k\rangle$. If $k = p$, this operator is known as projection operator.

Using Q, the introduced operator (6) can be represented as a real matrix

$$\langle x|\widehat{a}_{kp}|y\rangle = a_{kp}(x, y),$$

where $x, y \in Q$. It is not difficult to understand, that these matrices are basis images, considered above.

Using Z, we can present any single particle operator $\widehat{F}$ as follows

$$\widehat{F} = \sum_{k,p} |k\rangle\langle p|\langle k|\widehat{F}|p\rangle.$$

Operator $\widehat{F}$ can be written as a matrix, using Q and (5), then the right part of this equation takes the form (3). As a result we find that some of representations of single particle operators can be considered as basis grayscale images.

III. Wavelet Basis Images

Basis images can be generated by DWT. In calculation the DWT techniques do not use matrix methods and the basis wavelet images can be achieved by transform of standard basis.

Wavelet coefficients

The DWT coefficients have a block structure due to orthogonality of $N \times N$ matrix U. In case of single level transform this matrix consists of two parts, L and H, known as low and high frequency blocks.

Let G be a frequency representation of a $N \times N$ grayscale image

$$F = UGU^T,$$

$$G = U^T FU.$$

Applying the MATLAB notation, we write DWT as follows

$$G = dwt(F) = \begin{bmatrix}cA & cH\\ cV & cD\end{bmatrix}, \tag{7}$$

$$F = idwt(cA, cH, cV, cD).$$

The introduced blocks cA, cH, cV, and cD are approximation coefficients, horizontal, vertical and diagonal details or LL, LH, HL, and HH frequency bands. Four coefficient blocks are denoted by L and H frequency bands. Let assume that U^T is concatenated of these items that are $N \times N/2$ matrices

$$U^T = [L, H].$$

Then we find that

$$cA = LFL^T,$$
$$cH = LFH^T,$$
$$cV = HFL^T,$$
$$cD = HFH^T.$$

It results in the hierarchic decomposition of F as approximation and details

$$F = L^T(cA)L + L^T(cH)H + H^T(cV)L + H^T(cD)H.$$

The DWT coefficient matrix G can be considered as a three-dimensional array $G = \{G_{kpz}\}$ of size $N/2 \times N/2 \times 4$. Index $z = 1, 2, 3, 4$ labels the cA, cH, cV and cD blocks respectively, for example, $G_{kp1} = cA_{kp}$.

Block structure of basis

To calculate basis images we use equation (4)

$$a_{kp} = Ue_{kp}U^T.$$

According to (7) indexes (k, p) belong to one of the blocks cA, cH, cV or cD. Let $(k, p) \in cD$, so there is a set of basis items

$$a_{(kpD)} = idwt(O, O, O, e_{kp}), \qquad (8)$$

where O is a $N/2 \times N/2$ matrix of zeros. Here the upper indexes are in brackets to label number of the matrices instead of indicating the pixel position. In other words, we perform an orthogonal transformation of the unit block matrix

$$a_{(kpD)} \leftrightarrows \begin{bmatrix} O & O \\ O & e_{kp} \end{bmatrix}.$$

The total number of basis images $\{a_{(kpD)}\}$ is $N^2/4$, every image is a $N \times N$ matrix.

It is important to note that the equation (4) gives solution by MATLAB functions `dwt` and `idwt`. The reason is that in practice the DWT calculations are often based on the filter function techniques [24]. These techniques were developed for signal processing without referring to the orthogonal matrix U. Usually wavelets are introduced numerically or by recurrent equations so the calculation of U is a problem except, for example, the Haar wavelet.

Using the block coefficients cD and $a_{(kpD)}$ we can get an approximation of original image

$$D = \sum_{k,p} cD_{kp} a_{(kpD)}.$$

This image has diagonal details only.

The wavelet coefficient structure results in basis of four blocks. The blocks refer to cA, cH, cV, and cD similarly to (8)

$$\Big\{\{a_{(kpA)}\}, \{a_{(kpH)}\}, \{a_{(kpV)}\}, \{a_{(kpD)}\}\Big\}.$$

Every block has $N^2/4$ basis $N \times N$ images. As a result the representation over the wavelet basis images takes the form

$$F = \sum_{k,p} \Big(cA_{kp} a_{(kpA)} + cH_{kp} a_{(kpH)} + cV_{kp} a_{(kpV)} + cD_{kp} a_{(kpD)} \Big). \qquad (9)$$

The Fig. 1 shows basis images, $N = 6$, for the wavelet $db6$, i. e. the orthogonal Daubechies wavelet [25]. The basis set has four blocks, the blocks refer to approximation, horizontal, vertical and diagonal details. The first items of these blocks a_{11A}, a_{11H}, a_{11V}, and a_{11D} are presented in the Fig. 1.

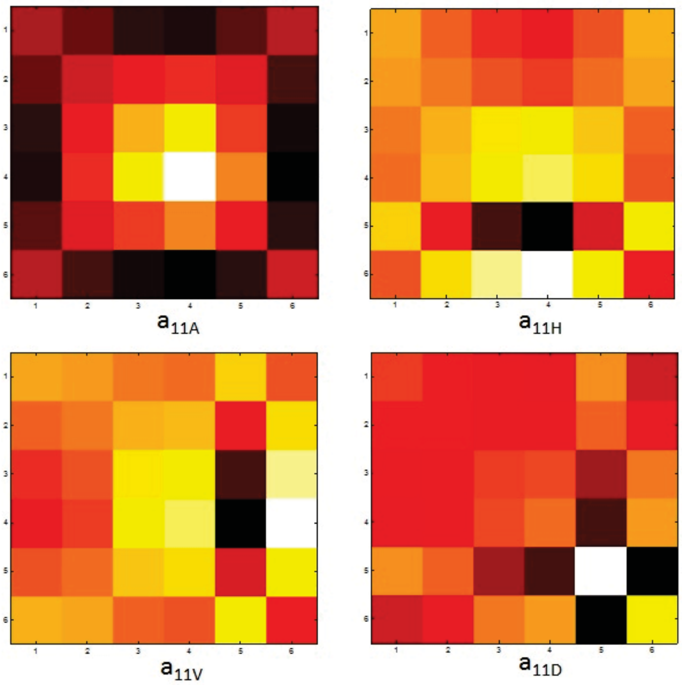

Fig. 1. Block structure of basis image for wavelet $db6$. There are four 6×6 matrices a_{11A}, a_{11H}, a_{11V} and a_{11D}. They belong to blocks that refer to approximation coefficients, horizontal, vertical and diagonal details

Basis image blocks structure

Indeed, the considered above function `dwt` can produce another basis. For this case in accordance with (7) every basis images has a block structure

$$d_{kp} = dwt(e_{xy}) = \begin{bmatrix} d_{(kpA)} & d_{(kpH)} \\ d_{(kpV)} & kp_{(kpD)} \end{bmatrix}.$$

Fig. 2 shows set of 36 images d_{kp}, $k, p = 1, \ldots, 6$, for wavelet $db6$. Every image has a block structure that corresponds to approximation, horizontal, vertical, and diagonal details.

Fig. 3 illustrates role of the blocks of items d_{kp}. At the top two images d_{55} and e_{55} are connected by DWT, wavelet $db6$. Image d_{55} has block structure shown at Fig. 2. A block of approximation, horizontal, vertical, and diagonal coefficients is discarded and original is retrieved. The original cannot be retrieved perfectly because of the discarded coefficients. Then there are artifacts that indicate which block was omitted. This is an illustration of role of blocks in visual perception.

A watermarking scheme

Orthogonality of basis images may be used for blind detection algorithm in watermarking techniques.

Consider an example based on representation (9). Let a message M be embedded into DWT coefficients, e. g. in cV_{kp}. The standard scheme has the following steps.

1) Select using secrete key two set of pixels Y_1 and Y_2, brightness of which will be changed or not after embedding.

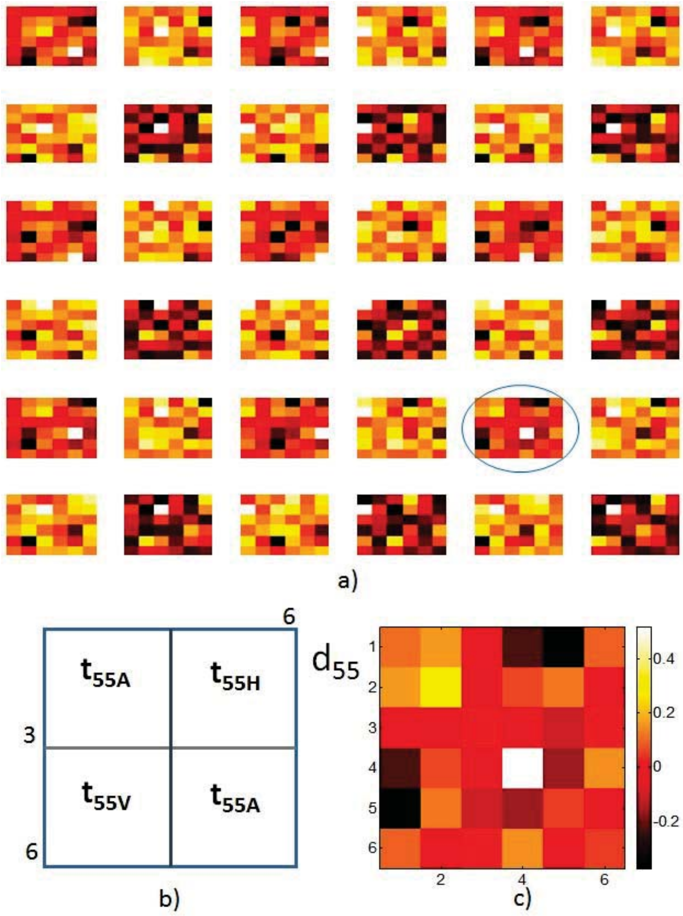

Fig. 2. Block structure of basis images d_{kp} for wavelet $db6$, $N = 6$. a)T h e set of 36 items, every item has block structure. b) and c) Four blocks of coefficients of image d_{55}

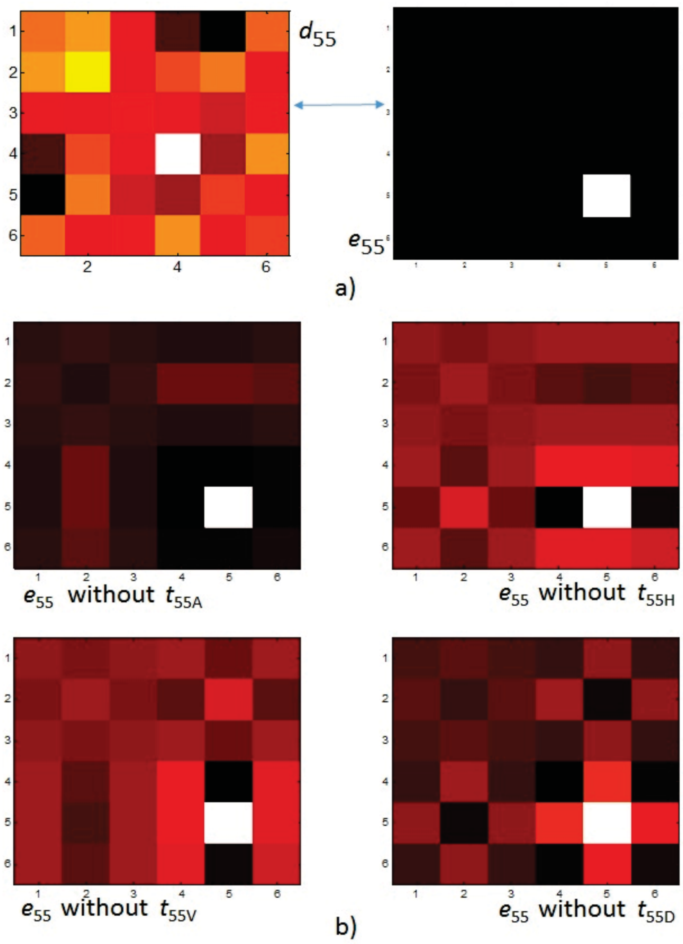

Fig. 3. Properties of the coefficient blocks for basis image. a) Image d_{55} is transformed from original image e_{55}. b) The retrieved original after removing from d_{55} one of the blocks: approximation coefficients, horizontal, vertical and diagonal details

2) Embed the message into Y_1 with the chosen algorithm $Y_M \leftarrow embed(M, Y_1, K)$, where K is a set of embedding parameters including the secrete key.
3) Extract data from frequency domain by detection algorithm.

New feature is that our detection algorithm is blind and it does not require the image in frequency domain. It works as follows. After embedding, the image F_M has a term $\sum_{k,p} Y_{Mkp} a_{(kpV)}$, that contains message. Detection algorithm does not need the initial image, it can extract the data using the equation $Y_{Mkp} = \langle F_M, a_{(kpV)} \rangle$.

IV. Color Basis Images

Digital color image is a three dimensional array that provides decomposition over basis images.

Orthogonal transformation of three-dimensional array

To transform a three-dimensional array T of size $M \times N \times Z$, it needs three orthogonal matrices U, V, and W of size $M \times M$, $N \times N$, and $Z \times Z$. Similarly to two-dimensional case (2) the transformation can be presented as a decomposition over tensor products of the matrix columns

$$T = \sum_{k,p,s} t_{kps} \, u_k \otimes \ v_p \otimes w_s,$$

where a set of three-dimensional arrays

$$a_{kps} = u_k \otimes \ v_p \otimes w_s,$$

is an orthonormal basis. In accordance with general properties the set can be achieved from the standard basis

$$U, V, W: \ e_{kps} \rightleftharpoons a_{kps},$$

where e_{kps} is a three-dimensional array, that has one non-zero element equal to 1 at position (k, p, s). Every item e_{kps} is a product of unit vectors

$$e_{kps} = e_k \otimes e_p \otimes e_s.$$

If $Z = 3$ the three dimensional array can describe a color image. The next observation is valid. Any tensor product of a matrix and a vector of three components has the form

$$A \otimes w = cat(3, Aw_1, Aw_2, Aw_3), \tag{10}$$

where cat is concatenation of matrices along dimension $d = 3$. Here we also use MATLAB notation. Operation cat is not commutative, in accordance with its definition. The three concatenated matrix array has the first matrix Aw_1, the second and the third Aw_2 and Aw_3 respectively. The representation (10) is valid if the product is replaced with the sum of products

$$A \otimes w \to \sum_u A_u \otimes w_u,$$

where all matrices have equal dimension and vector has three components. Equation (10) allows us to consider any three-dimensional array $M \times N \times 3$ as a color image.

Color images

Digital color image of RGB type can be described by three matrices R, G, and B often named Red, Green, and Blue channels. The matrices have equal size and consist of a three-dimensional array of size $M \times N \times 3$

$$C = cat(3, R, G, B).$$

From the definition it follows that the Red channel of the image C is the first matrix, $C_{mn1} = R_{mn}$, the Green and Blue channel is $C_{mn2} = G_{mn}$ and $C_{mn3} = B_{mn}$ respectively.

The color channel controls the display pixels brightness to visualize image. From (10) it follows that digital array $A \times w$ can be considered as a color image, which Red, Green, and Blue channels are Aw_1, Aw_2, and Aw_3.

Color of basis images

Consider an array of unit vectors from standard basis

$$e_{kps} = e_{kp} \otimes e_s,$$

where $e_{kp} = e_k \otimes e_p$ is a unit matrix and $s = 1, 2, 3$. These items can be represented as a color RGB images

$$e_{kps} = cat(3, e_{kp}\delta_{s1}, e_{kp}\delta_{s2}, e_{kp}\delta_{s3}).$$

The color can be find by the next procedure. For $s = 1$ we have

$$e_{kp1} = cat(3, e_{kp}, 0, 0).$$

It means that the color image e_{kp1} has a Red channel only. This is matrix e_{kp}, that has a non-zero pixel at position (k, p). Taking into account, that in RGB model the color $(0, 0, 0)$ is black, so image e_{kp1} looks as a one red pixel on a black background. Similarly, e_{kp2} and e_{kp3} look as one green and blue pixel on a black background.

Orthogonal transform of standard basis of unit vectors allows us to create another basis, for example,

$$a_{kps} = a_{kp} \otimes e_s,$$

where $s = 1, 2, 3$. The obtained items can also be considered as color images

$$a_{kps} = cat(3, a_{kp}\delta_{s1}, a_{kp}\delta_{s2}, a_{kp}\delta_{s3}).$$

Therefore, for $s = 1$ we have

$$a_{kp1} = cat(3, a_{kp}, 0, 0).$$

It follows that the item has a Red channel, it presented by matrix a_{kp}. From the image processing point of view, this matrix is a grayscale image and it does not have color. However, if the grayscale image is in Red channel only, it will be colored in shades of the red.

The considered colorizing can illustrate the pixel energy redistribution in orthogonal transform. That is one of the attractive properties for applications. In the case of basis sets e_{kp} and a_{kp} the next map is valid

$$cat(3, e_{kp}, 0, 0) \to cat(3, a_{kp}, 0, 0).$$

It means that a red pixel is transformed into an image in red shades and vise versa. Then we find two processes of the pixel energy scattering and concentration.

As a result, we find the decomposition of the RGB image over the orthonormal set of color basis images

$$C = cat(3, R, G, B) =$$
$$\sum_{k,p,s} \sigma_{kps} cat(3, a_{kp}\delta_{s1}, a_{kp}\delta_{s2}, a_{kp}\delta_{s3}) =$$
$$\sum_{k,p} \Big(cat(3, a_{kp}, 0, 0)\sigma_{kp\,1} + cat(3, 0, a_{kp}, 0)\sigma_{kp\,2} +$$
$$cat(3, 0, 0, a_{kp})\sigma_{kp\,3} \Big).$$

V. Conclusions

Orthogonal transformation has many attractive features for applications. Orthogonal transformation of digital images can be represented as a decomposition over basis matrices or basis images. We introduce grayscale and color basis images. For particular case of DWT found basis wavelet images have a block structure. We construct an example of detection algorithm for frequency embedding techniques. Moreover, introduced block representation leads to effective algorithms of block parallelization.

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