Algorithm of the Hybrid Transformation Method for Modeling Dynamic Systems

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***Abstract*—The paper solves the scientific problem of developing a method algorithm for the study of nonlinear dynamic systems. To increase the accuracy and speed of calculations, the paper proposes an algorithm for the hybrid method of transformations for the study of nonlinear mathematical models of a polynomial structure. The paper presents a method of polynomial transformations for the study of systems with three degrees of freedom, a study of a nonlinear vibration protection system with three degrees of freedom is carried out. The paper presents a hybrid numerical-analytical method for the analysis of nonlinear mathematical models of a general polynomial structure, which makes it possible to study systems with controlled accuracy while reducing the resource intensity of calculations. The method introduces additional complex exponential variables, formulas for calculating the transformation coefficients and the transformed system are presented. An analytical solution is constructed for the transformed system in the resonant and nonresonant cases. For the economical calculation of the right parts of the polynomial structure, formulas are presented and it is proposed to apply Pan's scheme with preliminary processing of the coefficients. The developed algorithm of the method of polynomial transformations makes it possible to construct an approximate analytical solution, taking into account the nonlinear components of higher degrees of the polynomial. The proposed algorithm of the method makes it possible to study the dynamic characteristics of the object under study, special cases of subharmonic, polyharmonic regimes, determine extreme regimes, and resonance with controlled accuracy. The above algorithm is implemented in the created software package using the modern object-oriented programming language C#. The proposed method makes it possible to carry out a qualitative and quantitative analysis of models of dynamic systems.**

I. INTRODUCTION

When operating various dynamic systems and mechanisms in real time, an urgent scientific problem is the creation of effective methods for analyzing normal and extreme operating modes of the studied technical objects. Therefore, it is necessary to develop methods and algorithms that allow solving the set scientific problem for a wide range of technical objects of a polynomial structure.

To construct an analytical solution, a simplification of the studied nonlinear equation is used, for example, the linearization of equations, which leads to significant errors in the model under study and incorrect formulation of the original modeling problem. For some particular cases, exact solutions of nonlinear equations are known, for example, for the Riccati and Bernoulli equations. However, most nonlinear models cannot be converted to equations with a known solution without losing the qualitative properties of the model. Exact solutions are not known for most nonlinear mathematical models of complex technical systems with many degrees of freedom.

Traditional approximate analytical methods, such as the method of a small parameter, method of Van Der Pol, method of Krylov–Bogolyubov, and averaging have a number of disadvantages and limitations that lead to qualitatively incorrect results of the analysis of nonlinear models [10,11]. It is necessary to develop approximate analytical methods and numerical methods of the required accuracy and low complexity for the analysis of nonlinear models of a general polynomial structure with constant and periodic parameters.

Section II provides an overview of recent work on systems analysis methods. Section III presents a transformation method for studying dynamical systems. Section IV presents the algorithm of the hybrid transformation method. Section V presents an assessment of the effectiveness of the hybrid transformation method. Section VI presents an estimate of the complexity of the transformation method. Section VII presents the results of the practical application of the method to the study of vibration protection systems. Section VIII CONCLUSION presents the main results of the work.

II. RELATED WORKS

We present a review of modern methods for the analysis of nonlinear dynamical systems in recent scientific papers. The work [1] presents two additive Runge-Kutta methods with fourth and fifth order accuracy. The methods are tested for the Van der Pol and Kaps problems on singular perturbations. The paper [2] proposes a two-stage method for fitting stiff models of ordinary differential equations to experimental data using polynomial approximation. The paper [3] proposes a new method for calculating second-order initial problems for ordinary differential equations using a nonlinear interpolation function. The paper [4] presents a numerical method for solving linear ordinary differential equations based on a posteriori quasi-Newtonian method. The paper [5] presents a new method for solving a wide class of problems involving ordinary and partial differential equations based on the spline collocation method. The work [6] proposes a quantum algorithm for linear ordinary differential equations, based on the so-called spectral methods, alternative to finite difference methods, which approximate the solution. In [7], the Robin method is studied as a development of the iterative Picard method for solving differential equations. In [8], an algorithm was proposed that allows one to find interval estimates for solutions with a given accuracy based on a kd-tree over a space formed by interval initial conditions for ordinary differential equations. In [9], a new implemented numerical method based on the scheme of the third-order Runge-Kutta method was proposed. When operating devices and mechanisms in real time, it is necessary to apply effective methods to calculate emerging extreme modes and eliminate them in a timely manner. Also, for industrial operation, processors with low performance are used. Therefore, methods should be used that allow efficient calculation of modes using optimal computing resources. The method of transformations proposed in the work allows one to determine such extreme regimes, to investigate subharmonic, polyharmonic regimes, and to determine resonance. Under subharmonic modes, free oscillations include harmonics, the frequencies for which are an integer number of times greater than the fundamental frequency. In polyharmonic modes, the harmonic force contains several harmonics. For technical systems, an extreme operating mode can occur during overload, start-up, acceleration, acceleration, and braking of the engine. In the extreme operating mode, abrupt changes in phase variables occur, which occur, for example, under the condition of the coincidence or multiplicity of the natural frequencies of the system and the frequency of the external disturbance. An extreme regime is a regime in which the rigidity of a system of ordinary differential equations increases. When solving such problems by standard numerical methods, the integration step should be reduced by an order of magnitude in order to identify jumps and points of a removable discontinuity of the first kind. For technical systems, an extreme operating mode occurs under adverse factors and conditions that go beyond the normal operating mode. The extreme mode for a nonlinear model of a technical system in many cases can be represented by subharmonic, superharmonic, polyharmonic oscillations, parametric and autoparametric resonances, relaxation or discontinuous oscillations. Among the traditional numerical methods for solving differential equations of a polynomial structure, the family of Runge–Kutta methods with an adaptive step is most used. The application of higher-order Runge–Kutta methods requires significant computational costs at each stage of calculations.

III. A TRANSFORMATION METHOD FOR STUDYING EXTREME AND STANDARD MODES OF DYNAMIC SYSTEMS

Let us write the studied nonlinear system of three second-order differential equations in matrix form:

$$I\ddot{q}+B\dot{q}+Cq=2H_1cos(\omega t)+2H_2sin(\omega t)+ \sum_{|\nu|=2}^{4}h_\nu cos(\omega t)^{\nu_1}sin(\omega t)^{\nu_2}q_1^{\nu_3}q_2^{\nu_4}q_3^{\nu_5}\dot{q}_1^{\nu_6}\dot{q}_2^{\nu_7}\dot{q}_3^{\nu_8}. \quad (1)$$

The non-linear parts for the three differential equations are represented by polynomials of the fourth degree. Let us assume that system (1) satisfies the conditions of Picard's theorem on the existence and uniqueness of a solution to the Cauchy problem for a system of ordinary differential equations. By the conditions of Picard's theorem, the right side of the system $F(t,Q)$ defined on the set $R=\{(t,Q)\,/\,|t-t_0|\le c_1; Q-Q_0=max|q_i-q_{i0}|\le c_2\}$, $F(t,Q)$ continuous on the set R for t,Q and for any (t,Q_1), (t,Q_2) set R the Lipschitz condition is satisfied Q: $F(t,Q_1)-F(t,Q_2)\le LQ_1-Q_2$, where L – Lipschitz constant.

The right nonlinear part of the system, including the second, third and fourth powers, in general, can contain 486 components defined by the vector index.

Here $q=[q_1,q_2,q_3]^T$ – vector of required variables, $Q=[q_1q_2q_3\dot{q}_1\dot{q}_2\dot{q}_3]^T$ – vector of phase variables, B,C – constant nondegenerate square matrices of the third order, I – identity matrix.

To write the non-linear part, a vector index with integer non-negative components is used:

$$\nu=(\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8),\ |\nu|=\nu_1+\nu_2+\ldots+\nu_8,\ 0\le\nu_i\le 4.$$

Constant column vectors with real coefficients are represented as: $H_1=[h_{11},h_{21},h_{31}]^T$, $H_2=[h_{12},h_{22},h_{32}]^T$, $h_\nu=[h^1{}_\nu,h^2{}_\nu,h^3{}_\nu]^T$

We assume that the characteristic matrix equation $Det[I\lambda^2+B\lambda+C]=0$ has complex conjugate roots $\lambda_s,\bar{\lambda}_s, s=3,4,\ldots,8$ with small negative real parts. We assume that the nonlinear components $|h_\nu|<1$ small.

We introduce complex conjugate variables to write periodic functions:

$$x_1=exp(i\omega t)\ \text{и}\ x_2=\bar{x}_1=exp(-i\omega t),\ \lambda_{1,2}=\pm i\omega.$$

We write the periodic functions in new variables:

$$cos(\omega t)=\frac{1}{2}(x_1+x_2)\ \text{и}\ sin(\omega t)=\frac{1}{2i}(x_1-x_2).$$

In accordance with the main stages of the method of polynomial transformations presented in the work of G.I. Melnikov [10], we will perform: bringing the system of nonlinear differential equations with constant and periodic parameters to normal form, linear transformation, polynomial transformation and solution of the autonomous transformed system.

Let us bring system (1) with the addition of complex conjugate variables to the normal form:

$$\dot{X}=PX+\sum_{|\nu|=2}^{4}p_\nu x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}x_4^{\nu_4}x_5^{\nu_5}x_6^{\nu_6}x_7^{\nu_7}x_8^{\nu_8} \quad (2)$$

where $X=[x_1x_2q_1q_2q_3\dot{q}_1\dot{q}_2\dot{q}_3]^T$.

Next, we perform a linear transformation:

$$Y=LX, \quad (3)$$

to reduce the linear part to a diagonal form:

$$\dot{Y}=\Lambda Y+\sum_{|\nu|=2}^{4}\tilde{p}_\nu y_1^{\nu_1}y_2^{\nu_2}y_3^{\nu_3}y_4^{\nu_4}y_5^{\nu_5}y_6^{\nu_6}y_7^{\nu_7}y_8^{\nu_8}. \quad (4)$$

At the stage of linear transformation, we find a nonsingular matrix L . Complex coefficients of the nonlinear part $\tilde{p}_\nu$ systems (4) are found by rearranging the terms in the sum (2) after a linear transformation (3).

Matrix Λ has a diagonal

$$\Lambda = diag\left[\lambda_1, \lambda_2, ..., \lambda_8\right] \quad (5)$$

with complex conjugate roots $\lambda_{2S} = \overline{\lambda}_{2S-1}, s = 1,..,4$

At the next stage, a polynomial transformation is performed up to fourth-order terms, inclusive:

$$y_S = z_S + \sum_{|\nu|=2}^{4} a_\nu^S z_1^{\nu_1} z_2^{\nu_2} z_3^{\nu_3} z_4^{\nu_4} z_5^{\nu_5} z_6^{\nu_6} z_7^{\nu_7} z_8^{\nu_8}, \; s = 3,\ldots,8, \quad (6)$$

where a_ν^S desired conversion coefficients.

The introduced additional complex conjugate variables are not transformed $y_S = z_S, s = 1,2$.

As a result of the polynomial transformation, we obtain a system of the form:

$$\dot{z}_S = \lambda_S z_S + \sum_{|\nu|=2}^{4} q_\nu^S z_1^{\nu_1} z_2^{\nu_2} z_3^{\nu_3} z_4^{\nu_4} z_5^{\nu_5} z_6^{\nu_6} z_7^{\nu_7} z_8^{\nu_8}, \quad (7)$$

where q_ν^S – the coefficients of the transformed system, which are calculated for special values of the vector index according to the iterative formula presented in [11].

Imagine the derivation of formulas for determining unknown coefficients. To shorten the notation, we introduce the notation:

$$Z^\nu = z_1^{\nu_1} z_2^{\nu_2} z_3^{\nu_3} z_4^{\nu_4} z_5^{\nu_5} z_6^{\nu_6} z_7^{\nu_7} z_8^{\nu_8}, \; Z = \left(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\right).$$

Let us represent the system (4) in the variables Z of the transformation (6).

$$\dot{y}_S = \lambda_S y_S + P_S\left(Y\right) = \lambda_S z_S + \lambda_S \sum_{|\nu|=2}^{4} a_\nu^S Z^\nu + P_S\left(Z\right),$$

when designating non-linear parts P_S.

Let's differentiate (6).

$$\dot{y}_S = \dot{z}_S + \sum_{|\nu|=2}^{4} a_\nu^S \left(Z^\nu\right)', \quad (8)$$

where $\left(Z^\nu\right)' = Z^\nu \sum_{k=1}^{8} \nu_k z_k^{-1} \dot{z}_k$.

Taking into account the formula for the transformed system (7), we represent the derivative.

$$\left(Z^\nu\right)' = Z^\nu \sum_{k=1}^{8} \nu_k z_k^{-1} \left(\lambda_k z_k + \sum_{|\nu|=2}^{4} q_\nu^k Z^\nu\right) = Z^\nu \sum_{k=1}^{8} \lambda_k \nu_k + Z^\nu \sum_{k=1}^{8} \nu_k z_k^{-1} \sum_{|\nu|=2}^{4} q_\nu^k Z^\nu.$$

Let us perform a substitution in (8) of expressions for the derivative and formula (7).

$$\dot{y}_S = \lambda_S z_S + \sum_{|\nu|=2}^{4} q_\nu^S Z^\nu + \sum_{|\nu|=2}^{4} (a_\nu^S Z^\nu \sum_{k=1}^{8} \lambda_k \nu_k) + \sum_{k=3}^{8} (\sum_{|\nu|=2}^{4} a_\nu^S Z^\nu \nu_k z_k^{-1} \sum_{|\mu|=2}^{4} q_\mu^k Z^\mu).$$

Equating the resulting expression for $\dot{y}_S$ in system (4) in variables Z we get the equality:

$$\lambda_S z_S + \lambda_S \sum_{|\nu|=2}^{4} a_\nu^S Z^\nu + P_S\left(Z\right) = \lambda_S z_S + \sum_{|\nu|=2}^{4} q_\nu^S Z^\nu + \sum_{|\nu|=2}^{4} (a_\nu^S Z^\nu \sum_{k=1}^{8} \lambda_k \nu_k) + \sum_{k=3}^{8} (\sum_{|\nu|=2}^{4} a_\nu^S Z^\nu \nu_k z_k^{-1} \sum_{|\mu|=2}^{4} q_\mu^k Z^\mu).$$

After simplifying the equality, we obtain formulas for determining unknown coefficients:

$$\sum_{|\nu|=2}^{4} q_\nu^S Z^\nu + \sum_{|\nu|=2}^{4} (a_\nu^S Z^\nu \left(\sum_{k=1}^{8} \lambda_k \nu_k - \lambda_S\right)) + \sum_{k=3}^{8} (\sum_{|\nu|=2}^{4} a_\nu^S Z^\nu \nu_k z_k^{-1} \sum_{|\mu|=2}^{4} q_\mu^k Z^\mu) = P_S\left(Z\right).$$

To determine the transformation coefficients and the transformed system, we equate the coefficients at the same powers Z, solve the system of algebraic equations.

In accordance with the method of polynomial transformations, the special values of the vector index at s are found as integer non-negative solutions of the equations:

$$\lambda_1 \nu_1 + \lambda_2 \nu_2 + \ldots + \lambda_8 \nu_8 - \lambda_s = 0,$$
$$\nu_1 + \nu_2 + \ldots + \nu_8 = 2,3,4, \quad s = 3,\ldots,8.$$

Instead of complex conjugate roots λ_i with small real parts, we take the imaginary parts for which we can write the equality:

$$\left(\nu_1 - \nu_2\right) Im\left(\lambda_1\right) + \left(\nu_3 - \nu_4\right) Im\left(\lambda_3\right) + \left(\nu_5 - \nu_6\right) Im\left(\lambda_5\right) + \left(\nu_7 - \nu_8\right) Im\left(\lambda_7\right) - Im\left(\lambda_s\right) = 0.$$

We equate the coefficients of the transformed system to zero for non-singular values of the vector index, and equate the coefficients of the transformation to zero for special values of the vector index. The coefficients of the transformed system are calculated at special values of the vector index, and the transformation coefficients are calculated at non-special values of the vector index.

Let's move on to new complex conjugate variables:

$$z_S = u_S exp\left(itIm\lambda_S\right), s = 3,..,8$$

The transformed system (7) in new variables is presented in an autonomous form:

$$\dot{u}_S = u_S Re\lambda_S + \sum_{|\nu|=2}^{4} q_\nu^S u_1^{\nu_1} u_2^{\nu_2} u_3^{\nu_3} u_4^{\nu_4} u_5^{\nu_5} u_6^{\nu_6} u_7^{\nu_7} u_8^{\nu_8}, \quad (9)$$

$$u_3 = \overline{u_4}, u_5 = \overline{u_6}, u_7 = \overline{u_8}$$

At the next stage, we solve the autonomous system.

Let us perform an exponential change of variables to pass to a system with real coefficients:

$$u_S = \rho_S exp\left(i\theta_S\right) \tag{10}$$

We represent system (9) as:

$$\begin{gathered}\dot{\rho}_S = \rho_S Re\lambda_S + \sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8} \; Re\left(q_\nu^S exp\left(Ae\right)\right), \\ \rho_S \dot{\theta}_S = \sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8} Im\left(q_\nu^S exp\left(Ae\right)\right), \\ Ae = i\left(\theta_1\left(\nu_1-\nu_2\right)+\ldots+\theta_7\left(\nu_7-\nu_8\right)-\theta_S\right).\end{gathered} \tag{11}$$

For finding ρ_S и θ_S it is necessary to solve a system of six differential equations for $s = 3,5,7$, because

$$u_{3,4} = \rho_3 exp\left(\pm i\theta_3\right), u_{5,6} = \rho_5 exp\left(\pm i\theta_5\right),$$

$$u_{7,8} = \rho_7 exp\left(\pm i\theta_7\right) \; .$$

The first two equations for $s = 1,2$ are the complement of the system by the introduced complex conjugate variables $z_{1,2} = exp\left(\pm i \quad t\right.$.

To determine the stationary solution, we equate the right-hand sides of (9) to zero and find the solution to the nonlinear system of algebraic equations:

$$\begin{gathered}u_S Re\lambda_S + \sum\nolimits_{|\nu|=2}^{4} q_\nu^S u_1^{\nu_1} u_2^{\nu_2} u_3^{\nu_3} u_4^{\nu_4} u_5^{\nu_5} u_6^{\nu_6} u_7^{\nu_7} u_8^{\nu_8} = 0 \\ , s = 3,5,\end{gathered} \tag{12}$$

In variables (10), we represent the system of equations for determining the stationary solution (12) in the form:

$$\begin{gathered}\rho_S Re\lambda_S = -\sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8} Re\left(q_\nu^S exp\left(Ae\right)\right), \\ \sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8} Im\left(q_\nu^S exp\left(Ae\right)\right) = 0, \\ Ae = i\left(\theta_1\left(\nu_1-\nu_2\right)+\ldots+\theta_7\left(\nu_7-\nu_8\right)-\theta_S\right).\end{gathered} \tag{13}$$

Given the decomposition of the exponent into trigonometric functions in the form:

$$\begin{gathered}\left(Re\left(f\right)+iIm\left(f\right)\right)\exp\left(ig\right) = \\ Re\left(f\right)cos\left(g\right) - Im\left(f\right)sin\left(g\right) + \;, \\ iIm\left(f\right)cos\left(g\right) + iRe\left(f\right)sin\left(g\right)\end{gathered}$$

Let's represent the transformed autonomous system (11) in the form:

$$\begin{gathered}\dot{\rho}_S = \rho_S Re\lambda_S + \sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8}\left(RI\right), \\ RI = Re\left(q_\nu^S\right)cos\left(Ae\right) - Im\left(q_\nu^S\right)sin\left(Ae\right), \\ Ae = \left(\theta_1\left(\nu_1-\nu_2\right)+\ldots+\theta_7\left(\nu_7-\nu_8\right)-\theta_S\right),\end{gathered} \tag{14}$$

$$\rho_S \dot{\theta}_S = \sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8}\left(Im\left(q_\nu^S\right)cos\left(Ae\right) + Re\left(q_\nu^S\right)sin\left(Ae\right)\right),$$

To determine the stationary solution, we equate the right-hand sides of (14) to zero:

$$\rho_S Re\lambda_S = \tag{15}$$

$$\begin{gathered}= \sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8}\left(Re\left(q_\nu^S\right)cos\left(Ae\right) - Im\left(q_\nu^S\right)sin\left(Ae\right)\right), \\ \sum\nolimits_{|\nu|=2}^{4} \rho_1^{\nu_1+\nu_2} \ldots \rho_7^{\nu_7+\nu_8}\left(Im\left(q_\nu^S\right)cos\left(Ae\right) + Re\left(q_\nu^S\right)sin\left(Ae\right)\right) = 0.\end{gathered}$$

Considering that the first two equations in the system for $s = 1,2$ not converted $z_1 = exp\left(it\omega\right)$ and the form of new variables $z_S = \rho_S exp\left(itIm\lambda_S + i\theta_S\right)$ we get the equalities:

$$\rho_S = 1, \theta_S = 0 \text{ at } s = 1,2 .$$

Let us write the autonomous system of differential equations (14) taking into account the equalities $\rho_{1,2} = 1, \theta_{1,2} = 0$ in the following form:

$$\dot{\rho}_S = \rho_S Re\lambda_S + \tag{16}$$

$$\begin{gathered}+\sum\nolimits_{|\nu|=2}^{4} \rho_3^{\nu_3+\nu_4} \rho_5^{\nu_5+\nu_6} \rho_7^{\nu_7+\nu_8}\left(Re\left(q_\nu^S\right)cos\left(Ae\right) - Im\left(q_\nu^S\right)sin\left(Ae\right)\right), \\ \rho_S \dot{\theta}_S = \sum\nolimits_{|\nu|=2}^{4} \rho_3^{\nu_3+\nu_4} \rho_5^{\nu_5+\nu_6} \rho_7^{\nu_7+\nu_8}\left(Im\left(q_\nu^S\right)cos\left(Ae\right) + Re\left(q_\nu^S\right)sin\left(Ae\right)\right),\end{gathered}$$

Having obtained the solution of the autonomous system (16), we substitute ρ_S and θ_S into the formula

$$z_S = \rho_S exp\left(itIm\lambda_S + i\theta_S\right), s = 3,..,8 \; .$$

By the transformation formulas (6) we find

$$y_S = z_S + \sum\nolimits_{|\nu|=2}^{4} a_\nu^S Z^\nu \; .$$

To represent the solution in the original variables, we perform the inverse linear replacement: $X = L^{-1}Y$.

The resulting autonomous system of differential equations in general form (16) has a certain form depending on the special vector indices determined for the initial system.

Let us define special vector indices in the nonresonant case, when the natural oscillation frequencies of the system and the frequency of external forces do not coincide and are not multiples.

At q_ν^3 find special indexes

ν : (00100011),(00101100),(00210000),(11100000).

At q_ν^5 find special indexes

ν : (00001011),(00002100),(00111000),(11001000).

At q_ν^7 find special indexes

ν : (00000021),(00001110),(00110010),(11000010).

In the absence of resonances, the transformed autonomous system (16) has the form:

$$\dot{\rho}_S = \rho_S Re\lambda_S + \sum_{|\nu|=2}^{4} Re\left(q_\nu^S\right)\rho_3^{\nu_3+\nu_4}\rho_5^{\nu_5+\nu_6}\rho_7^{\nu_7+\nu_8}, \tag{17}$$

$$\rho_S\dot{\theta}_S = \sum_{|\nu|=2}^{4} Im\left(q_\nu^S\right)\rho_3^{\nu_3+\nu_4}\rho_5^{\nu_5+\nu_6}\rho_7^{\nu_7+\nu_8}, s = 3,5,7$$

Substituting into system (17) the coefficients q_ν^S with special indexes found, we obtain six differential equations of the first order. In this case, all special indexes are different for $s = 3,5,7$, therefore, to simplify the writing of the coefficients, we omit q_ν^S upper index s.

$$\dot{\rho}_3 = \rho_3 Re\lambda_3 + \rho_3\rho_7^2 Req_{0,0,1,0,0,0,1,1} + \rho_3\rho_5^2 Req_{0,0,1,0,1,1,0,0} + \rho_3^3 Req_{0,0,2,1,0,0,0,0} + \rho_3 Req_{1,1,1,0,0,0,0,0},$$

$$\rho_3\dot{\theta}_3 = \rho_3\rho_7^2 Imq_{0,0,1,0,0,0,1,1} + \rho_3\rho_5^2 Imq_{0,0,1,0,1,1,0,0} + \rho_3^3 Imq_{0,0,2,1,0,0,0,0} + \rho_3 Imq_{1,1,1,0,0,0,0,0},$$

$$\dot{\rho}_5 = \rho_5 Re\lambda_5 + \rho_5\rho_7^2 Req_{0,0,0,0,1,0,1,1} + \rho_5^3 Req_{0,0,0,0,2,1,0,0} + \rho_3^2\rho_5 Req_{0,0,1,1,1,0,0,0} + \rho_5 Req_{1,1,0,0,1,0,0,0},$$

$$\rho_5\dot{\theta}_5 = \rho_5\rho_7^2 Imq_{0,0,0,0,1,0,1,1} + \rho_5^3 Imq_{0,0,0,0,2,1,0,0} + \rho_3^2\rho_5 Imq_{0,0,1,1,1,0,0,0} + \rho_5 Imq_{1,1,0,0,1,0,0,0},$$

$$\dot{\rho}_7 = \rho_7 Re\lambda_7 + \rho_7^3 Req_{0,0,0,0,0,0,2,1} + \rho_5^2\rho_7 Req_{0,0,0,0,1,1,1,0} + \rho_3^2\rho_7 Req_{0,0,1,1,0,0,1,0} + \rho_7 Req_{1,1,0,0,0,0,1,0},$$

$$\rho_7\dot{\theta}_7 = \rho_7^3 Imq_{0,0,0,0,0,0,2,1} + \rho_5^2\rho_7 Imq_{0,0,0,0,1,1,1,0} + \rho_3^2\rho_7 Imq_{0,0,1,1,0,0,1,0} + \rho_7 Imq_{1,1,0,0,0,0,1,0}.$$

Let us find a stationary solution in the nonresonant case, equating the right-hand sides in the equations to zero and solving the system of equations, we obtain a steady state. As a result, the author received a solution in the form:

$$\rho_3^2 = \frac{k13k22k30 - k12k23k30 - k13k20k32 + k10k23k32 + k12k20k33 - k10k22k33}{-k13k22k31 + k12k23k31 + k13k21k32 - k11k23k32 - k12k21k33 + k11k22k33}$$

$$\rho_5^2 = \frac{k13k21k30 - k11k23k30 - k13k20k31 + k10k23k31 + k11k20k33 - k10k21k33}{k13k22k31 - k12k23k31 - k13k21k32 + k11k23k32 + k12k21k33 - k11k22k33}$$

$$\rho_7^2 = \frac{k12k21k30 - k11k22k30 - k12k20k31 + k10k22k31 + k11k20k32 - k10k21k32}{-k13k22k31 + k12k23k31 + k13k21k32 - k11k23k32 - k12k21k33 + k11k22k33} \tag{18}$$

$$\theta_3 = t\left(\rho_7^2 Imq_{0,0,1,0,0,0,1,1} + \rho_5^2 Imq_{0,0,1,0,1,1,0,0} + \rho_3^2 Imq_{0,0,2,1,0,0,0,0} + Imq_{1,1,1,0,0,0,0,0}\right)$$

$$\theta_5 = t\left(\rho_7^2 Imq_{0,0,0,0,1,0,1,1} + \rho_5^2 Imq_{0,0,0,0,2,1,0,0} + \rho_3^2 Imq_{0,0,1,1,1,0,0,0} + Imq_{1,1,0,0,1,0,0,0}\right)$$

$$\theta_7 = t\left(\rho_7^2 Imq_{0,0,0,0,0,0,2,1} + \rho_5^2 Imq_{0,0,0,0,1,1,1,0} + \rho_3^2 Imq_{0,0,1,1,0,0,1,0} + Imq_{1,1,0,0,0,0,1,0}\right)$$

With the following renaming:

$k10 = Re\lambda_3 + Req_{1,1,1,0,0,0,0,0}$, $k13 = Req_{0,0,1,0,0,0,1,1}$, $k12 = Req_{0,0,1,0,1,1,0,0}$, $k11 = Req_{0,0,2,1,0,0,0,0}$, $k20 = Re\lambda_5 + Req_{1,1,0,0,1,0,0,0}$, $k23 = Req_{0,0,0,0,1,0,1,1}$, $k22 = Req_{0,0,0,0,2,1,0,0}$, $k21 = Req_{0,0,1,1,1,0,0,0}$, $k30 = Re\lambda_7 + Req_{1,1,0,0,0,0,1,0}$, $k33 = Req_{0,0,0,0,0,0,2,1}$, $k32 = Req_{0,0,0,0,1,1,1,0}$, $k31 = Req_{0,0,1,1,0,0,1,0}$.

We assume that the expression in the denominator is not equal to zero. To find the transformed system in the resonant case, when the natural oscillation frequencies of the system and the frequency of external forces coincide and are multiples, we define special indices. By definition, resonance is a sharp increase in the amplitude of forced oscillations when the external (exciting) frequency coincides with the internal (natural) frequency of the system.

Let us consider the case when the natural oscillation frequency of the system, corresponding to the first root, coincides with the frequency of the external periodic force. We define the following special vector indices.

At q_ν^3 find special indexes ν :

(00100011),(10000011),(00101100),(10001100),(00210000),

(10110000),(20010000),(01200000),(11100000),(21000000).

At q_ν^5 find special indexes ν :

(00001011),(00002100),(00111000),(10011000),(01101000), (11001000).

At q_ν^7 find special indexes ν :

(00000021),(00001110),(00110010),(10010010),(01100010), (11000010).

In this case, the transformed autonomous system (16) is represented by six first-order differential equations. Solving the system, we obtain a steady state in the case of resonance. The result is a solution of the form:

$$\rho_3 = \sqrt{\frac{k33}{k34}}, \rho_5 = \sqrt{\frac{k35}{k36}}, \rho_7 = \sqrt{\frac{k37}{k38}}, \tag{19}$$

$$\theta_3 = \arccos\left(\frac{k30}{\sqrt{k31}\sqrt{k32}}\right) \tag{20}$$

$$\theta_5 = \rho_7^2 Imq_{0,0,0,0,1,0,1,1} + \rho_5^2 Imq_{0,0,0,0,2,1,0,0} + \rho_3^2 Imq_{0,0,1,1,1,0,0,0} + \frac{k30}{\sqrt{k31}\sqrt{k32}}\rho_3 Imq_{0,1,1,0,1,0,0,0} + \frac{k30}{\sqrt{k31}\sqrt{k32}}\rho_3 Imq_{1,0,0,1,1,0,0,0} + Imq_{1,1,0,0,1,0,0,0} + sin[\theta_3]\rho_3 Req_{0,1,1,0,1,0,0,0} - sin[\theta_3]\rho_3 Req_{1,0,0,1,1,0,0,0},$$

$$\theta_7 = \rho_7^2 Imq_{0,0,0,0,0,0,2,1} + \rho_5^2 Imq_{0,0,0,0,1,1,1,0} + \rho_3^2 Imq_{0,0,1,1,0,0,1,0} + \frac{k30}{\sqrt{k31}\sqrt{k32}}\rho_3 Imq_{0,1,1,0,0,0,1,0} + \frac{k30}{\sqrt{k31}\sqrt{k32}}\rho_3 Imq_{1,0,0,1,0,0,1,0} + Imq_{1,1,0,0,0,0,1,0} + sin[\theta_3]\rho_3 Req_{0,1,1,0,0,0,1,0} - sin[\theta_3]\rho_3 Req_{1,0,0,1,0,0,1,0}$$

Using the method of polynomial transformations, the original non-linear system of three second-order differential equations with constant and periodic parameters is converted to an autonomous form up to fourth-order terms.

To economically calculate the right-hand sides of a polynomial structure, Pan's scheme with coefficient preprocessing, shown below, can be applied. Consider the traditional methods for computing polynomials [12]. William George Horner's universally accepted method for calculating polynomials involves only n–1 multiplications and n additions. To calculate a polynomial of the n-th degree, Horner's scheme is presented in the form:

$$f(x)=\left(...\left(\left(\left(x+a_1\right)\cdot x+a_2\right)\cdot x+a_3\right)...+a_{n-1}\right)\cdot x+a_n \qquad (21)$$

Let us present a generalization of the traditional Horner scheme for a polynomial in many variables of the form

$$f(x_1,\dots,x_m) = \sum_{i_1,\dots,i_m=0}^{n} a_{i_1,\dots,i_m}\prod_{j=1}^{m} x_j^{i_j}, \sum_{j=1}^{m} i_j \le n .$$

$$f(x_1,\dots,x_m) = a_0 + \sum_{i_1=1}^{m} x_{i_1}\left(a_{i_2} + \sum_{i_2=i_1}^{m} x_{i_2}\left(a_{i_1,i_2} + \sum_{i_3=i_2}^{m} x_{i_3}\left(a_{i_1,i_2,i_3} + \dots\right)\right)\right)$$

In the works of J. Todd [13], another scheme was proposed for calculating a polynomial of the sixth degree using auxiliary polynomials:

$$p_1(x)=(x+b_1)x, \quad p_2(x)=(x+p_1+b_2)(p_1+b_3), \quad f(x)=(p_2+b_4)(p_1+b_5)+b_6 \qquad (22)$$

To determine the coefficients b_i solve a system of equations. To compute a sixth-degree polynomial in one variable according to Todd's scheme, three multiplications and seven divisions are required.

In accordance with another scheme for calculating polynomials, presented in the works of Yu.L. Ketkov to calculate the polynomial of the nth degree for $n > 5$ necessary $\frac{n+1}{2}+\frac{n}{4}$ multiplications and $n+1$ additions.

Methods for economical calculation of polynomials with preprocessing of coefficients are presented in the works of V.Ya. Pan [14]. In accordance with the two-stage Pan scheme, to calculate the n-th degree polynomial, it is necessary $\frac{n}{2}+1$ multiplications and $n+1$ additions. For the Pan scheme, auxiliary polynomials are used:

$$p_0 = x^2, p_1 = x + b_1, p_{4s+1} = p_{4s-3}\left(\left(p_0 + x + b_{4s-2}\right)\left(p_0 + b_{4s-1}\right) + b_{4s}\right) + b_{4s+1},$$

$$p_{4k+3} = p_{4k+1}\left(p_0 + b_{4k+2}\right) + b_{4k+3}, \quad s = 1,\dots,k$$

$$f_n(x) = a_0 p_n \text{ , при } n = 4k+1, 4k+3;$$

$$f_n(x) = a_0 x p_{n-1} + a_n \text{, при } n = 4k+2, 4k+4.$$

We present a generalization of V. Ya. Pan's scheme with preliminary processing of the coefficients for the polynomial $f_n(x_1,\dots,x_m)$ from many variables.

$$p_0 = \sum_{j_1=1}^{m} x_{j_1}\sum_{j_2=j_1}^{m} x_{j_2}, p_1 = \sum_{j=1}^{m} x_j + b_1,$$

$$p_{4s+1} = p_{4s-3}\left(\left(\sum_{j=1}^{m} x_j + p_0 + b_{4s-2}\right)\left(p_0 + b_{4s-1}\right) + b_{4s}\right) + b_{4s+1},$$

$$p_{4k+3} = p_{4k+1}\left(p_0 + b_{4k+2}\right) + b_{4k+3}, \quad s = 1,\dots,k$$

$$f_n = a_0 p_n \text{ , at } n = 4k+1, 4k+3;$$

$$f_n = \sum_{j=1}^{m} a_{0j} x_j p_{n-1} + a_n \text{ , at } n = 4k+2, 4k+4.$$

The works of V. Ya. Pan [15] present several schemes with preprocessing for calculating polynomials with real and complex coefficients, in which the number of multiplications is halved.

The two-stage economical Pan calculation scheme with preprocessing of coefficients makes it possible to halve the number of multiplication operations when calculating polynomials, which leads to a significant increase in performance when used in iterative schemes.

The introduction of the method of economical calculation of polynomials in method schemes can be effectively used to solve the Cauchy problem with nonlinearities of a polynomial structure in order to increase the productivity of calculations.

IV. ALGORITHM OF THE HYBRID TRANSFORMATION METHOD FOR INVESTIGATION OF DYNAMIC SYSTEMS

The hybrid method is based on the analytical method of polynomial transformations, the fourth order numerical Runge-Kutta method, the economical calculation of Pan polynomials, Newton's method for solving systems of nonlinear algebraic equations (SNAE). The combination of analytical and numerical methods makes it possible to combine their advantages and completely solve the problem. Consider the Cauchy problem for systems of nonlinear ordinary differential equations.

Let the functions $f_i\left(cos(\omega t), sin(\omega t), q_1, q_2, ..., q_m\right)$ $i = 1,2,...,m$ defined in the form of degree polynomials n f $t \in [t_0, t_k]$, $(q_1, q_2, ..., q_m) \in R^m$, $\omega \in R$.

Required to define functions $q_1(t), q_2(t), ..., q_m(t)$ which are the solution of the system of nonlinear differential equations on the interval $[t_0, t_k]$:

$$\begin{cases} q'_1(t) = f_1(cos(\omega t), sin(\omega t), q_1, q_2, ..., q_m), \\ q'_2(t) = f_2(cos(\omega t), sin(\omega t), q_1, q_2, ..., q_m), \\ q'_m(t) = f_m(cos(\omega t), sin(\omega t), q_1, q_2, ..., q_m), \end{cases} \quad (23)$$

and satisfy the initial conditions

$q_1(t) = q_{01}, q_2(t) = q_{02}, ..., q_m(t) = q_{0m}$, where $q_{01}, q_{02}, ..., q_{0m}$, - given real numbers.

Let us assume that the conditions of Picard's theorem on the existence and uniqueness of a solution to the Cauchy problem are satisfied for a system of differential equations. The right side of the system is defined, continuous, and satisfies the Lipschitz condition.

Provided that the functions can be represented in the form of polynomials in variables:

$$f_m(cos(\omega t), sin(\omega t), q_1, q_2, ..., q_m) = \sum_{\nu_1 + ... + \nu_{m+2} = 1}^{n} d_{\nu_1, ..., \nu_{m+2}} cos^{\nu_1}(\omega t) sin^{\nu_2}(\omega t) q_1^{\nu_3} ... q_m^{\nu_{m+2}}$$

Let us present the main stages of the hybrid transformation method [16] for solving nonlinear systems of differential equations with a polynomial structure.

Stage 1. Let us represent the system in matrix form (24), using the vector index $\nu = [\nu_1 \nu_2 ... \nu_m]$ with integer non-negative components.

Denote $|\nu| = \nu_1 + \nu_2 + ... + \nu_m$ - the sum of the components of the vector index.

$$\dot{Q} + CQ = H_1 cos(\omega t) + H_2 sin(\omega t) + \sum_{|\nu|=2}^{n} H_\nu cos^{\nu_1}(\omega t) sin^{\nu_2}(\omega t) q_1^{\nu_3} ... q_m^{\nu_{m+2}} \quad (24)$$

$Q = [q_1, q_2, ..., q_m]^T$ – column vector of sought functions,

$\dot{Q} = [\dot{q}_1, \dot{q}_2, ..., \dot{q}_m]^T$ – derivative column vector,

C – constant nonsingular square matrix $m \times m$,

Constant column vectors are represented as:

$$H_1 = [h_{11}, h_{21}, \ldots, h_{m1}]^T, H_2 = [h_{12}, h_{22}, \ldots, h_{m2}]^T$$

$H_\nu = [h^1{}_\nu, h^2{}_\nu, ..., h^m{}_\nu]^T$ – column vector of small $|h^k{}_\nu| < 1$ nonlinear coefficients.

Stage 2. At the second stage, we represent the system in normal form. Recording in normal form is performed with the addition of the system with new variables

$x_1 = exp(i\omega t)$ and $x_2 = \overline{x}_1 = exp(-i\omega t)$

$$\dot{X} = N \cdot X + R(X), \quad (25)$$

where $X = [x_1, x_2, q_1, ..., q_m]^T$ - column vector of sought functions with complemented variables. The square matrix N is obtained by complementing the matrix C:

$$N = \begin{bmatrix} i\omega & 0 & 0 \\ 0 & -i\omega & 0 \\ (H_1 - iH_2)/2 & (H_1 + iH_2)/2 & C \end{bmatrix},$$

$$R = \left[0, 0, \sum_{|\nu|=2}^{n} h_\nu^1 x_1^{\nu_1} x_2^{\nu_2} q_1^{\nu_3} ... q_m^{\nu_{m+2}}, ..., \sum_{|\nu|=2}^{n} h_\nu^m x_1^{\nu_1} x_2^{\nu_2} q_1^{\nu_3} ... q_m^{\nu_{m+2}}, \right]^T$$

Stage 3. For system (25), we solve the characteristic equation $Det[\Lambda - N] = 0$ and define conjugate roots $\lambda_S, s = 1, .., m+2$ with small negative real parts.

Define a nonsingular diagonal matrix $\Lambda = diag[\lambda_1, ..., \lambda_{m+2}]$.

We check the presence of external and internal resonances by comparing the roots and external frequency ω.

Performing a linear transformation:

$$Y = L \cdot X \quad (26)$$

The nondegenerate matrix of the linear transformation L is determined from the matrix equation $L \cdot N = \Lambda \cdot L$

The result of transformation (26) is a system of the form:

$$\dot{Y} = \Lambda \cdot Y + R(Y) \quad (27)$$

Right nonlinear part $R(Y)$ rewritten after a linear change of variables $R(Y) = R(L \cdot X)$

Stage 4. Special values of the vector index $\nu = [\nu_1 \nu_2 ... \nu_{2m+2}]$ are found by solving equations:

$$\lambda_1 \nu_1 + \ldots + \lambda_{m+2} \nu_{m+2} - \lambda_s = 0,$$
$$\nu_1 + \nu_2 + \ldots + \nu_{m+2} = 2, 3, ..., n, s = 1, 2, ..., m+2$$

The special values of the vector index uniquely determine the shape of the transformed autonomous system. Only non-linear components with special indices remain in the autonomous system.

Stage 5 We carry out a polynomial transformation for system (27) of the form:

$$y_s = z_s + \sum_{|\nu|=2}^{n} \left(a_\nu^s z_1^{\nu_1} z_2^{\nu_2} ... z_{m+2}^{\nu_{m+2}} \right), \ s = 1, 2, ..., m+2 \quad (28)$$

The result of transformation (28) is the system:

$$\dot{z}_S = \lambda_s z_s + \sum_{|\nu|=2}^{n} \left(p_\nu^s z_1^{\nu_1} z_2^{\nu_2} ... z_{m+2}^{\nu_{m+2}} \right), \; s = 1,2,...,m+2 \tag{29}$$

where q_ν^s – coefficients of the transformed system, a_ν^s - conversion factors.

Stage 6 Performing the definition of conversion coefficients a_ν^s и p_ν^s of the transformed system (29) according to the formulas:

$$\sum_{|\nu|=2}^{n} q_\nu^S z_1^{\nu_1} z_2^{\nu_2} ... z_{m+2}^{\nu_{m+2}} + \sum_{|\nu|=2}^{n} \left(a_\nu^S z_1^{\nu_1} z_2^{\nu_2} ... z_{m+2}^{\nu_{m+2}} \left(\sum_{k=1}^{m+2} \lambda_k \nu_k - \lambda_S \right) \right) +$$

$$\sum_{k=3}^{m+2} . \sum_{|\nu|=2}^{n} \left(a_\nu^S z_1^{\nu_1} z_2^{\nu_2} .. z_{m+2}^{\nu_{m+2}} \nu_k z_k^{-1} \sum_{|\mu|=2}^{n} q_\mu^k z_1^{\mu_1} z_2^{\mu_2} ... z_{m+2}^{\mu_{m+2}} \right) = R_S,$$

Stage 7. To reduce to an autonomous form, we perform in the system (29) the transition to new complex variables:

$$z_S = u_S exp\left(itIm\lambda_S \right), s = 1,2,...,m+2 \tag{30}$$

The transformed system in new variables has form:

$$\dot{u}_S = u_S Re\lambda_S + \sum_{|\nu|=2}^{4} q_\nu^S u_1^{\nu_1} u_2^{\nu_2} ... u_{m+2}^{\nu_{m+2}}, \tag{31}$$

Stage 8. We turn to an autonomous system with real variables. For (31), we perform an exponential change of variables:

$$u_S = \rho_S exp\left(i\theta_S \right) \tag{32}$$

As a result, we obtain an autonomous system in general form:

$$\dot{\rho}_S = \rho_S Re\lambda_S + \sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} \dots \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Re\left(q_\nu^S exp\left(iU \right) \right),$$
$$U = \sum_{l=1}^{m+2} \theta_{2l-1} \left(\nu_{2l-1} - \nu_{2l} \right) - \theta_S \tag{33}$$
$$\rho_S \dot{\theta}_S = \sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} \dots \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Im\left(q_\nu^S exp\left(iU \right) \right).$$

In the nonresonant case, when the natural oscillation frequencies of the system and the frequency of external forces do not coincide and are not multiples, the arguments at the exponent are equal to zero, and the autonomous system has a simpler form.

$$\dot{\rho}_S = \rho_S Re\lambda_S + \sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} ... \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Re\left(q_\nu^S \right),$$
$$\rho_S \dot{\theta}_S = \sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} ... \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Im\left(q_\nu^S \right) \tag{34}$$

Stage 9 To determine the stationary solution, we equate the right parts of the autonomous system (10) to zero. Let us find the steady state by solving a system of nonlinear algebraic equations of the form:

$$\sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} \dots \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Re\left(q_\nu^S exp\left(iU \right) \right) = -\rho_S Re\lambda_S,$$
$$\sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} \dots \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Im\left(q_\nu^S exp\left(iU \right) \right) = 0, \tag{35}$$
$$U = \sum_{l=1}^{m} \theta_{2l-1} \left(\nu_{2l-1} - \nu_{2l} \right) - \theta_S.$$

To solve a system of nonlinear algebraic equations, we use Newton's method. To apply Newton's method, it is necessary to fulfill the following conditions: the functions of the left parts of the system of algebraic equations must be bounded, smooth, continuously differentiable, the first derivatives of the functions are uniformly separated from zero, the second derivatives of the functions must be uniformly bounded, the Jacobian matrix of the system of functions is nonsingular.

In the nonresonant case, in the system of nonlinear algebraic equations (34), the exponential powers are equal to zero, and system (35) has a simpler form:

$$\sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} ... \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Re\left(q_\nu^S \right) = -\rho_S Re\lambda_S,$$
$$\sum_{|\nu|=2}^{n} \rho_1^{\nu_1+\nu_2} ... \rho_{m+1}^{\nu_{m+1}+\nu_{m+2}} Im\left(q_\nu^S \right) = 0, \; s = 1,2,...,m+2$$

10 stage. To determine the transient mode of an autonomous system, we use the traditional numerical Runge–Kutta method with the introduction of the method of economical calculation of polynomials into the iterative scheme. The classical four-stage Runge-Kutta method has the fourth order of accuracy. Given initial conditions $q_1(t) = q_{01}, q_2(t) = q_{02}, ..., q_m(t) = q_{0m}$ transform to new variables, sequentially performing transformations (3),(5),(7),(9). The transformed autonomous system (33) with initial conditions defines the Cauchy problem in new variables. In the systems of nonlinear differential equations under study, the right-hand sides are polynomials in variables. For economical calculation of nonlinear parts, we use highly efficient and productive algorithms for calculating polynomials. The commonly accepted Horner method for computing polynomials involves n–1 multiplications and n additions. Methods for economical calculation of polynomials with preprocessing of coefficients are presented in the works of V.Ya. Pan. In accordance with the two-stage Pan scheme, to calculate the n-th degree polynomial, it is necessary $n/2+1$ multiplications and n+1 additions. A two-stage economical Pan calculation scheme with preprocessing of coefficients makes it possible to halve the number of multiplication operations when calculating polynomials. This leads to a significant increase in performance when applied in iterative schemes.

11 stage. We perform transformation to original variables:

$$z_S = \rho_S exp\left(itIm\lambda_S + i\theta_S \right), \; y_s = z_s + \sum_{|\nu|=2}^{n} \left(a_\nu^s z_1^{\nu_1} z_2^{\nu_2} ... z_{m+2}^{\nu_{m+2}} \right).$$

To represent the result in the original variables, we perform the inverse linear replacement: $X = L^{-1} \cdot Y$.

The hybrid transformation method makes it possible to analyze nonlinear systems with a polynomial structure and obtain the main dynamic characteristics of the systems.

V. EVALUATION OF THE EFFICIENCY OF THE ALGORITHM OF THE NUMERICAL-ANALYTICAL TRANSFORMATION METHOD

Let us evaluate the efficiency of the algorithm of the numerical-analytical method of transformations in the calculation of extreme modes. The efficiency of an algorithm is directly related to computational resources, the main of which are the execution time and memory used. When calculating by standard numerical Runge-Kutta methods to determine the extreme regime, it is necessary to determine the beginning of the occurrence of such a regime and apply an adaptive step, reducing it in the region of the occurrence of the extreme regime by tens of times.

Let us estimate the number of input parameters and operations performed in the calculation of extreme modes by the standard fourth-order Runge–Kutta method and the transformation method. The calculation model is represented by a system of second-order differential equations of a polynomial structure in the general form of a fourth-degree polynomial with all possible coefficients at the powers. The number of algorithm operations depends on the number of differential equations being solved in the system, on the number of components in the nonlinear parts of the equations, and on the number of calculated nodal points.

Table I presents estimates of the number of operations performed in the calculation of extreme modes for systems from two to six non-linear differential equations of the second order by the method of transformations on a section with 10 nodal points. The evaluation assumes the presence in the nonlinear parts of the equations of all components up to the fourth powers inclusive. The estimates are made on the basis of a computational experiment.

TABLE I. ESTIMATES OF THE NUMBER OF OPERATIONS FOR THE TRANSFORMATION METHOD

Number equations	2	3	4	5	6
Number operations	10080	83160	443520	1801800	6054048

Table II shows the number of input parameters and operations performed in the calculation of extreme modes for systems of two to six non-linear second-order differential equations with all non-linear components of polynomials by the transformation method and the fourth-order Runge–Kutta numerical method on a section with 10 nodal points. Estimating the execution time of an algorithm is an important factor in measuring the effectiveness of an algorithm. For industrial applications in the systems under consideration, processors of average performance are used - megaflops (the number of floating point operations per second). For example, peak double precision performance for Intel Pentium III 450MHz processors is 440 megaflops, for Intel Celeron M 900MHz processors - 690 megaflops, for Intel Pentium III-S 1GHz processors - 900 megaflops, for AMD C-50 processors - 860 megaflops, for processors ARMv7l 1GHz - 360 megaflops.

TABLE II. ESTIMATES OF THE NUMBER OF INPUT PARAMETERS AND OPERATIONS FOR THE TRANSFORMATION METHOD AND THE FOURTH-ORDER RUNGE–KUTTA METHOD

Number of equations in the system	**Number of input parameters**	**Number of operations for the transformation method**	**Number of operations for the Runge–Kutta 4 method**
2	408	10080	100800
3	1461	83160	369600
4	3964	443520	1108800
5	9040	1801800	2882880
6	18276	6054048	6726720

Table III shows the estimates of the execution time of the method algorithms on an Intel Celeron M 900 MHz processor of average performance when calculating the extreme mode in a section with 100 nodal points in the presence of all nonlinear components.

TABLE III. ESTIMATES OF THE ALGORITHM EXECUTION TIME (IN SECONDS) WHEN CALCULATING THE EXTREME MODE ON THE INTEL CELERON M 900 MHZ PROCESSOR

Models	**Transform method**	**Runge–Kutta 4 method**	**Time reduction (%)**
vibration protection system	0.001	0.021	95
anti-vibration system for derricks	0.001	0.025	96
robotic manipulators	0.002	0.031	93

When calculating nonlinear systems with three degrees of freedom in the presence of all nonlinear components of polynomials, the calculation time on an Intel Celeron M 900 MHz average processor in a section with 100 nodal points for the transformation method is 1 ms, and for the Runge-Kutta method it was 25 ms. with a comparative accuracy of the fourth order.

VI. EVALUATION OF THE COMPLEXITY OF ALGORITHMS

A comparative assessment of the complexity of the algorithms of the methods in calculations for extreme modes is carried out. Evaluation of the complexity of calculations is necessary to compare the speed of algorithms and determine the execution time, the amount of memory depending on the size of the data being processed. To assess the complexity of the algorithm, the complexity function is determined - the relationship between the input data and the number of algorithm operations. Let us consider the number of basic mathematical operations of the algorithm of the transformation method when calculating the stationary mode. The number of operations in the method depends on the number of differential equations being solved in the system, on the degrees of polynomials of the nonlinear parts of the equations, and on the number of calculated nodal points. The number of mathematical operations of the transformation method algorithm is estimated $O\left(\frac{m(m+n+2)!}{n!(m-1)!}\right)$, for the algorithm of the Runge–Kutta method of the fourth order, when calculating on an interval with k nodal points, it is estimated $O\left(\frac{4k(m+n+2)!}{n!m!}\right)$. When calculating the stationary mode by the transformation method for 10 nodal points of a system of four equations with right-hand sides in the form of a polynomial of the sixth degree, we reduce the calculation time by 60% compared to the calculation by the fourth-order Runge–Kutta numerical method. When calculating the extreme regime of a nonlinear model, the transformation method has a complexity that is less in comparison with the Runge–Kutta method with a comparative accuracy of the fourth order. The transformation method makes it possible to determine extreme modes, reduce the level of complexity and the time of performing calculations.

VII. INVESTIGATION OF NONLINEAR VIBRATION PROTECTION SYSTEMS WITH THREE DEGREES OF FREEDOM

In various industries, shock absorbers and dampers are used to reduce external influences. The shock absorber performs the absorption of part of the energy of external perturbing forces. The vibration protection system protects the object from external influences [17]. When creating a new vibration protection scheme, it is necessary that the natural frequencies of the system be significantly lower than the frequencies of external disturbing forces. In modern production, a huge number of types of shock absorbers are widely used, which differ in elastic, damping, vibration isolating and shockproof properties. The traditional method of vibration protection is to install a shock absorber or damper between the object of protection and the source of disturbances. The works [18,19] consider various mathematical models of vibration protection systems, which include non-linear shock absorbers or dampers. Consider a vibration protection system with three degrees of freedom, which includes parallel and series installation of non-linear shock absorbers or dampers. Vibration protection object with mass m_2 installed by means of non-linear shock absorbers and dampers through intermediate platforms with a mass of m_1 и m_3 . We believe that the nonlinear characteristics of shock absorbers and dampers are represented by polynomials of the third degree. The choice of a cubic characteristic is justified by the design features of the vibration isolators and the properties of the material of the elastic elements. In vibration isolators, conical springs with non-linear characteristics are used, as well as rubber elements with non-linear material properties. Similar schemes of vibration protection systems, but with linear characteristics, were considered in [20,21]. Using the method of transformations, we study a nonlinear mathematical model of a vibration protection system for various operating modes. We believe that the elastic and damping characteristics of the vibration protection system are represented by a cubic form. For the elastic force of the spring $kq+pq^3$ at $p>1$ the spring is assumed to be rigid. Here k,p - stiffness coefficients, c,d - damping coefficients. Under the action of external periodic forces, the base experiences vertical vibrations $f(t)=h_1 sin(\omega t)+h_2 cos(\omega t)$ with frequency ω . Dimensions of quantities: $\omega[c^{-1}], t[c], h[cм]$. When compiling the dynamic equations of the vibration protection system, the Lagrange equations. The system of equations of motion of the vibration protection system has the form:

$$\begin{aligned}
&m_1 q_1''(t)+c_1 q_1'(t)+c_2\left(q_1'(t)-q_2'(t)\right)+\\
&c_5\left(q_1'(t)-q_3'(t)\right)+d_1 q_1'(t)^3+d_2\left(q_1'(t)-q_2'(t)\right)^3+\\
&d_5\left(q_1'(t)-q_3'(t)\right)^3+k_1 q_1(t)+k_2\left(q_1(t)-q_2(t)\right)+\\
&k_5\left(q_1(t)-q_3(t)\right)+p_1 q_1(t)^3+p_2\left(q_1(t)-q_2(t)\right)^3+\\
&p_5\left(q_1(t)-q_3(t)\right)^3=c_1\left(h_1\omega\cos(t\omega)-h_2\omega\sin(t\omega)\right)+\\
&k_1\left(h_1\sin(t\omega)+h_2\cos(t\omega)\right),
\end{aligned}$$

$$\begin{aligned}
&m_2 q_2''(t)+c_2\left(q_2'(t)-q_1'(t)\right)+c_3\left(q_2'(t)-q_3'(t)\right)+\\
&d_2\left(q_2'(t)-q_1'(t)\right)^3+d_3\left(q_2'(t)-q_3'(t)\right)^3+k_2\left(q_2(t)-q_1(t)\right)+\\
&k_3\left(q_2(t)-q_3(t)\right)+p_2\left(q_2(t)-q_1(t)\right)^3+p_3\left(q_2(t)-q_3(t)\right)^3=0,
\end{aligned}$$

$$\begin{aligned}
&m_3 q_3''(t)+c_3\left(q_3'(t)-q_2'(t)\right)+c_4 q_3'(t)+c_5\left(q_3'(t)-q_1'(t)\right)+\\
&d_3\left(q_3'(t)-q_2'(t)\right)^3+d_4 q_3'(t)^3+d_5\left(q_3'(t)-q_1'(t)\right)^3+\\
&k_3\left(q_3(t)-q_2(t)\right)+k_4 q_3(t)+k_5\left(q_3(t)-q_1(t)\right)+\\
&p_3\left(q_3(t)-q_2(t)\right)^3+p_4 q_3(t)^3+p_5\left(q_3(t)-q_1(t)\right)^3=\\
&c_4\left(h_1\omega cos(t\omega)-h_2\omega sin(t\omega)\right)+k_4\left(h_1 sin(t\omega)+h_2 cos(t\omega)\right)
\end{aligned}$$

where q_1, q_2, q_3 – displacement relative to the equilibrium position of the system. Here the time derivatives are denoted

$$q_i'(t)=\dot{q}_i=dq_i/dt .$$

A similar system with a linear characteristic of elastic and damping elements was studied in [22]. Let's apply the transformation method. We write the system of nonlinear differential equations (SNDE) in matrix form:

$$\begin{aligned}
&A\ddot{q}+B\dot{q}+Cq=H_1 cos(\omega t)+H_2 sin(\omega t)+\\
&\sum_{|\nu|=2}^{4} h_\nu cos(\omega t)^{\nu_1} sin(\omega t)^{\nu_2} q_1^{\nu_3} q_2^{\nu_4} q_3^{\nu_5} \dot{q}_1^{\nu_6} \dot{q}_2^{\nu_7} \dot{q}_3^{\nu_8},
\end{aligned}$$

$$A=diag\left[m_1,m_2,m_3\right],$$

$$B=\begin{bmatrix} c_1+c_2+c_5 & -c_2 & -c_5 \\ -c_2 & c_2+c_3 & -c_3 \\ -c_5 & -c_3 & c_3+c_4+c_5 \end{bmatrix}, H_1=\begin{bmatrix} h_2 k_1+c_1 h_1\omega \\ 0 \\ h_2 k_4+c_4 h_1\omega \end{bmatrix},$$

$$C=\begin{bmatrix} k_1+k_2+k_5 & -k_2 & -k_5 \\ -k_2 & k_2+k_3 & -k3 \\ -k_5 & -k_3 & k_3+k_4+k_5 \end{bmatrix}, H_2=\begin{bmatrix} h_1k_1-c_1h_2\omega \\ 0 \\ h_1k_4-c_4h_2\omega \end{bmatrix}.$$

We represent the SNDE in normal form: $\dot{X}=NX+R(x)$.

Recording in normal form is performed with the addition of the system with new variables $x_1=exp(i\omega t)$, $x_2=exp(-i\omega t)$ and the representation in terms of them of the periodic coefficients: $cos(\omega t)=0.5(x_1+x_2)$ и $sin(\omega t)=-0.5i(x_1-x_2)$. Performing a linear transformation: $Y=LX$, define a diagonal matrix $\Lambda=diag[\lambda_1,...,\lambda_8]$. We verify that the characteristic matrix equation $Det[A\lambda^2+B\lambda+C]=0$ has complex conjugate roots $\lambda_s,\bar{\lambda}_s, s=3,\ldots,8$ with small negative real parts. The result of a linear transformation is a system of the form: $\dot{Y}=\Lambda Y+R(y)$. We check the equality and multiplicity of the roots $Im\lambda_s, s=1,\ldots,8$ to determine the presence of resonances. We determine the special values of the index for a fixed s as integer non-negative solutions of the equations:

$$\sum_{k=1}^{8}\nu_k Im(\lambda_k)-\lambda_S=0, \sum_{k=1}^{8}\nu_k=2,3,4, \quad s=3,\ldots,8 .$$

We perform the determination of coefficients a_ν^s and p_ν^s

$$\sum_{|\nu|=2}^{4}q_\nu^S Z^\nu+\sum_{|\nu|=2}^{4}(a_\nu^S Z^\nu\left(\sum_{k=1}^{6}\lambda_k\nu_k-\lambda_S\right))+$$
$$\sum_{k=3}^{6}(\sum_{|\nu|=2}^{4} a_\nu^S Z^\nu \nu_k z_k^{-1}\sum_{|\mu|=2}^{4}q_\mu^k Z^\mu)=R_S(Z).$$

We carry out the transformation: $y_s=z_s+\sum_{|\nu|=2}^{4}\left(a_\nu^s Z^\nu\right)$.

The result of the transformation is the system:

$$\dot{z}_S=\lambda_s z_s+\sum_{|\nu|=2}^{4}\left(p_\nu^s Z^\nu\right).$$

We perform the transition to new complex conjugate variables: $z_S=u_S exp(itIm\lambda_S)$.

The system in new variables has an autonomous form:

$$\dot{u}_S=u_S Re\lambda_S+\sum_{|\nu|=2}^{4}q_\nu^S U^\nu .$$

We turn to an autonomous system with real variables. We perform an exponential change of variables: $u_S=\rho_S exp(i\theta_S)$. As a result, we obtain an autonomous system in general form:

$$\dot{\rho}_S=\rho_S Re\lambda_S+\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Re\left(q_\nu^S exp(Ae)\right),$$
$$\rho_S\dot{\theta}_S=\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Im\left(q_\nu^S exp(Ae)\right),$$
$$Ae=i\left(\theta_1(\nu_1-\nu_2)+\ldots+\theta_7(\nu_7-\nu_8)-\theta_S\right).$$

In the nonresonant case, the arguments at the exponent are zero, and the autonomous system has a simpler form:

$$\dot{\rho}_S=\rho_S Re\lambda_S+\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Re\left(q_\nu^S\right),$$
$$\rho_S\dot{\theta}_S=\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Im\left(q_\nu^S\right).$$

We determine the steady state by solving a system of nonlinear algebraic equations of the form:

$$\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Re\left(q_\nu^S exp(Ae)\right)=-\rho_S Re\lambda_S,$$
$$\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Im\left(q_\nu^S exp(Ae)\right)=0.$$

In the nonresonant case, we solve a system of nonlinear algebraic equations of the form:

$$\rho_S Re\lambda_S+\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Re\left(q_\nu^S\right)=0,$$
$$\sum_{|\nu|=2}^{4}\rho_1^{\nu_1+\nu_2}...\rho_7^{\nu_7+\nu_8}Im\left(q_\nu^S\right)=0.$$

Stationary solutions are determined in the case of the absence of resonances (18) and in the case of resonance (19). To obtain a solution in the original variables, we perform the transformation: $z_S=\rho_S exp(itIm\lambda_S+i\theta_S)$, $y_s=z_s+\sum_{|\nu|=2}^{4}\left(a_\nu^s Z^\nu\right)$

To represent the result in the original variables, we perform the replacement, the inverse of the linear: $X=L^{-1}Y$. We will study the system with the following parameters:

$m_1=7$; $m_2=151$; $m_3=8$; $\omega=2.6; h_1=0.03; h_2=0.04;$
$c_1=0.01; c_2=0.011; c_3=0.012; c_4=0.013; c_5=0.014;$
$d_1=0.001; d_2=0.001; d_3=0.001; d_4=0.001; d_5=0.001;$
$k_1=14.312; k_2=13.717; k_3=12.114; k_4=11.192; k_5=10.317;$
$p_1=0.011; p_2=0.012; p_3=0.013; p_4=0.014; p_5=0.015.$

Let us represent the SNDE in normal form: $\dot{X}=NX+R(x)$. Performing a linear transformation: $Y=LX$. As a result of a linear transformation, we obtain a system with a linear diagonal matrix $\dot{Y}=\Lambda Y+R(y)$. Eigenvalues have small negative real parts. The steady polyharmonic mode of oscillations of the system has the form:

$$q_1=-0.0348cos(0.304t)-0.0018cos(1.982t)+0.0041cos(578t)-$$
$$0.1586cos(6t)+0.0153sin(0.304t)-0.0009sin(1.982t)+$$
$$0.0164sin(578t)+0.2539sin(6t),$$
$$q_2=-0.0687cos(0.304t)-0.0017cos(1.982t)-0.0015cos(578t)+$$
$$0.0115cos(6t)+0.0302sin(0.304t)-0.0013sin(1.982t)+$$
$$0.0140sin(578t)+0.0025sin(6t),$$
$$q_3=-0.0362cos(0.304t)-0.0030cos(1.982t)-0.0021cos(578t)+$$
$$0.045cos(6t)+0.0159sin(0.304t)-0.0016sin(1.982t)+$$
$$0.0047sin(578t)-0.0661sin(6t).$$

For the considered nonlinear vibration protection system, the efficiency depends on the selected parameters, on the amplitudes and frequencies of the external action. The studied vibration protection system with a nonlinear cubic characteristic is most effective in the case of external disturbance frequencies significantly higher than the natural frequencies of the system. As a result, a vibration protection system with three degrees of freedom, represented by a system of three non-linear differential equations with a non-linearity of the third degree, is considered. The object of vibration protection is installed by means of shock absorbers and dampers between two platforms, to which external periodic impact is transmitted. Non-linear shock absorbers and dampers have cubic characteristics. The method of transformations is applied to determine the steady state and resonant modes.

VIII. CONCLUSION

The paper solves the scientific problem of developing high-performance algorithms of numerical-analytical methods for the study of standard and extreme modes of operation of nonlinear dynamic systems. The paper presents the main methods for studying nonlinear systems of differential equations used in the modern theory of nonlinear mathematical modeling. To increase the accuracy and speed of calculations, the paper proposes an algorithm for the hybrid method of transformations for the study of nonlinear mathematical models of a polynomial structure. The paper presents a method of polynomial transformations for the study of systems with three degrees of freedom, a study of a nonlinear vibration protection system with three degrees of freedom is carried out. The paper proposes a hybrid numerical-analytical method for the analysis of nonlinear mathematical models of a general polynomial structure, which makes it possible to study systems with controlled accuracy while reducing the resource intensity of calculations. The method introduces additional complex exponential variables, formulas for calculating the transformation coefficients and the transformed system are presented. An analytical solution is constructed for the transformed system in the resonant and nonresonant cases. For the economical calculation of the right parts of the polynomial structure, formulas are presented and it is proposed to apply Pan's scheme with preliminary processing of the coefficients. The developed algorithm of the method of polynomial transformations makes it possible to construct an approximate analytical solution, taking into account the nonlinear components of higher degrees of the polynomial. The proposed algorithm of the method makes it possible to study the dynamic characteristics of the object under study, special cases of subharmonic, polyharmonic regimes, determine extreme regimes, and resonance with controlled accuracy. The above algorithm is implemented in the created software package using the modern object-oriented programming language C#. The proposed algorithmic formulas for the method of polynomial transformations make it possible to study nonlinear dynamic systems. Using the algorithm of the method allows one to study systems of a polynomial structure with right-hand sides in the form of polynomials in phase variables with constant, as well as with periodic parameters. The proposed algorithm of the transformation method and new algorithmic formulas make it possible to determine extreme modes, efficiently use computing resources, increase computational performance and reduce the level of complexity of calculations. The proposed hybrid method, which combines analytical and numerical methods, makes it possible to conduct a qualitative and quantitative analysis of nonlinear dynamic systems with a polynomial structure and obtain the main characteristics of dynamic systems.

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