Spline-based Image Interpolators and Decimators: Design and Efficient Implementation

Karen Egiazarian and Atanas Gotchev
Department of Signal Processing, Tampere University of Technology
Korkeakoulunkatu 1, Tampere, Finland
{karen.egiazarian, atanas.gotchev}@tut.fi

Abstract

This tutorial paper addresses the problem of high-quality resizing that is arbitrary-scale conversion of images cast as projection of the underlying function on spline-like spaces. Such spaces are determined by the corresponding bases, which fully determine the quality of scale conversion. In our setting, the space-generating functions are chosen from the class of piecewise (spline-based) basis functions of minimal support constructed as combinations of uniform *B*-splines of different degrees. The benefit of such functions is that they allow for various effective optimization strategies and are also very efficient in practical implementation. Specifically, we present a minimax optimization technique for designing effective reconstruction functions specified in Fourier domain and demonstrate its performance by comparing with other designs in terms of error kernel behavior and by interpolation experiments with real images. We present also two efficient schemes for interpolation and decimation respectively based on the designed reconstruction kernels and comment on practical implementation issues.

Index Terms: B-splines, spline spaces, least squares projection.

I. INTRODUCTION

Digital images are created by discrete sensors such as CCD or CMOS, and are represented in the form of discrete pixels on rectangular grids. Many image processing applications, such as zooming in and out, affine transforms, unwarping, rotation, require generating new pixels at coordinates different from the given grid. These new pixels can be generated by a two-stage procedure. First, a certain continuous function is reconstructed to fit the given uniformly sampled data and then to so-reconstructed function is resampled at the desired new coordinates. In this process, the reconstruction (often called also interpolation) functions play a crucial role.

In a general framework, the reconstruction functions are considered as generators of shift-invariant spaces. Thus, the sought continuous function can be represented by a linear combination of the reconstruction (basis) functions weighted with proper weights. This framework allows for making a relation between the continuous function to be sampled or reconstructed and its approximation (projection) on a shift-invariant space in terms of its discrete representation (samples). For shift-invariant spaces, the approximation theory provides tools for quantifying the error between the function and its approximation.

Among reconstruction functions generating shift-invariant spaces, *B*-splines have attracted wide attention due to their good approximation properties and efficient practical implementation. Next stage in development of spline-like basis functions has involved piecewise polynomial functions expressed as linear combinations of *B*-splines of different degrees. Having such a combination allows for optimization attempts to achieve even better approximation properties for shortest possible function's support that is for precisely the same computational efficiency as in the case of regular B-splines.

In this talk, we address the problem of image interpolation and decimation by optimized spline-based kernels. We begin by stating the interpolation problem in signal and frequency domains. We then describe the most general distortions caused by non-ideal interpolation and quantitative measures for evaluating the performance of a particular interpolator. The central part of the talk is devoted to the design and analysis of piecewise-polynomial basis functions of minimal support which can be used in image scale-conversion tasks. The design is based on linear combinations of uniform *B*-splines of different degrees. We compare the basis kernels designed by different optimization criteria in terms of error kernel behavior and by interpolation experiments with real images. We then present two efficient schemes for practical implementation of the considered functions. The first scheme targets efficient interpolation while the second one is based on near least squares projection and targets efficient decimation.

II. SPLINE-LIKE BASIS FUNCTIONS

Most commonly used image reconstruction models are based on convolution [1], [2]. For such models, the sought continuous function is represented as a linear combination of basis functions [3]. Analytical tools for evaluating the function performance have been developed based on the approximation theory [4], [5].

In this section, we demonstrate how to design and analyze piecewise polynomial basis functions obtained as a linear combination of *B*-splines. What makes the *B*-splines preferable for interpolation is their efficiency, since they can achieve a desired approximation order L with a minimum support of L. Lth-order spline-based interpolators can be constructed as combinations of a *B*-spline of degree $L-1$ with its derivatives. The resulting functions have been called *splines of minimal support* [6] or MOMS (maximal order minimal support) [7]. They have been optimized to have the smallest possible approximation constant in order to achieve minimal asymptotic approximation error. Functions of this class have been reported as superior for image interpolation [7], [8]

Another approach has also adopted the use of uniform *B*-splines of different degrees as linearly combined bases for forming piecewise polynomial interpolators [9, 12]. In this approach, the weighting parameters in the combination have been tuned by an optimization approach inspired by the digital filter design rather than the approximation theory. The design aim has been put on approximating the ideal band-limited interpolator in frequency domain by clustering the frequency response zeros around the multiples of the sampling rate ($2k\pi$ in terms of angular frequency). In contrast with the Lth-order interpolators where $L-1$ zeros are placed exactly at $2k\pi$, this earlier approach [12] sacrifices the multiplicity, distributing the zeros among the stopband to provide better stopband attenuation.

We present practical cases of optimized combinations of *B*-splines of third- and first-degree *B*-splines and of fifth-, third-, and first degree *B*-splines. These cases provide a good way to deal with both smooth and sharp image areas and are attractive from the computational complexity point of view. For these cases two optimization techniques are presented and compared: the first one is based on approximation theoretic assumptions and the second one operates in frequency domain based on classical filter design assumptions. By the latter technique, the adjusted parameters are optimized in such a way that the resulting interpolation function sufficiently suppresses imaging frequencies [9]. As a result, the obtained piecewise kernels achieve better frequency characteristics than the classical *B*-splines, that is, flatter in the passband, steeper in the transition band, and with sufficient attenuation in the stopband. The technique developed in [7] leads to the design of so-called O-MOMS functions. The main difference between the two optimized families is in the position of frequency response zeros around the multiples of the sampling rate ($2k\pi$ in terms of angular frequency). MOMS functions have multiple zeros at those points complying with the Strang-Fix conditions for Lth-order approximation functions. In the alternative design, the Lth order of approximation is traded for a better frequency domain stopband performance. A minimax optimization procedure leads to *clustering* the zeros around $(2k\pi)$, thus forming the desired stopbands. Thus, a good compromise is reached between the increased interpolation capabilities for relatively high frequencies and the controllable low frequency error. Experiments have proved the best interpolation capabilities of the functions thus constructed, especially for preserving high-frequency image details [9].

II. EFFICIENT IMPLEMENTATIONS

Efficient filtering structure of piecewise-polynomial basis functions for an arbitrary-scale interpolation is based on the so-called Farrow structure [10] (Figure 1). In the shown structure, the only input parameter besides the input image to be interpolated, is the fraction interval determining the distance of the new pixel to be generated from the previous-given pixel position.

It has been shown [11], that for the case of image decimation, the Farrow structure can be modified to perform a near LS operation (a hybrid between continuous and discrete LS) as shown in Figure 2.

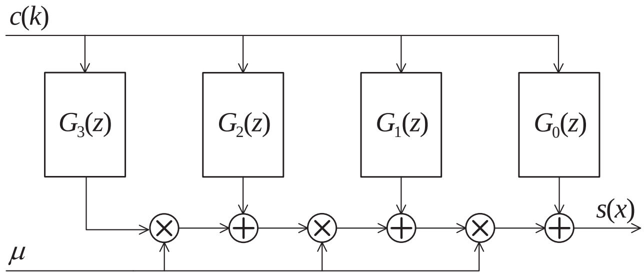

Figure 1. Efficient interpolation by piecewise-polynomial kernels. See [9] for details.

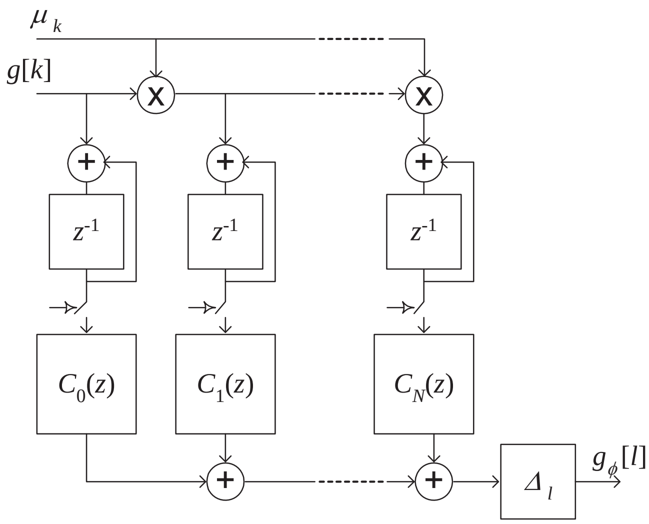

Figure 2. Efficient decimation by near LS projection on piecewise-polynomial (spline-like) space. See [10] for details.

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